Capacity of Fox's H-function Fading Channel with Adaptive Transmission

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Novel closed-form exact expressions for the capacity of the recently-introduced Fox's H-function fading channel are derived. The expressions are obtained assuming different types of adaptive transmission strategies, namely, channel inversion with fixed rate (CIFR), truncated channel inversion with fixed rate (TIFR), optimum rate adaptation with fixed power (ORA) and optimum power and rate adaptation (OPRA). The obtained expressions are versatile and can be used to evaluate the capacity for virtually any fading channel one might be interested in as a special case.

Introduction: Considerable efforts have recently been devoted to model the fading phenomenon using appropriate statistical distributions. This resulted in new generalized models such as α-μ, κ-μ and η-μ, among many others. Also, versatile composite models that describe both small-scale and large-scale fading have been proposed. This includes the Extended generalized-K (EGK), which was propsoed to describe the power of the received signal in millimeter wave and free-space optical channels. Another important model is the Fox's H-function distribution, which subsumes most, if not all, of the known fading distributions [1, Tables II-V]. In fact, the experimental results reported in [2] show that this distribution represents an excellent fit for fading in vehicle-to-vehicle (V2V) communication more than any other model.

In this Letter, we study the capacity of Fox's H-function fading channel assuming the different adaptive transmission strategies proposed in [3]. The new expressions in this work provide a unified form, which can handle several of the well-known simple and composite fading environments as special cases. Moreover, they are given in terms of the Fox-H function, for which simple MATLAB implementations have been recently proposed in lots of works, e.g., in [4].

Fox's H-function Fading: We consider communications over a fading channel where the SNR, γ, follows the Fox's H-function probability density function (PDF) given by [5, Eq. (1)]

$$f_\gamma(\gamma) = \kappa H_{p,q}^{m,n}\left(\lambda\gamma \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right), \quad \gamma > 0 \tag{1}$$

where $\lambda > 0$ and κ are constants such that $\int_0^\infty f_\gamma(\gamma)d\gamma = 1$ and $(x_j, y_j)_\ell$ is a shorthand for $(x_1, y_1), \dots, (x_\ell, y_\ell)$. The H-function itself is defined as

$$H_{p,q}^{m,n}\left(\zeta \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right)\zeta^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)}\zeta^{-s}ds \tag{2}$$

where the path of the integration $\mathcal{L}$ goes from $\sigma - i\infty$ to $\sigma + i\infty$, $\sigma \in \mathbb{R}$ such that all the poles of $\Gamma(b_j + B_j s), j = 1, 2, \dots, m$ are separated from those of $\Gamma(1 - a_j - A_j s), j = 1, 2, \dots, n$. One important property of the H-function that will be used in this work is its Mellin transform defined as $\mathcal{M}\{f_\gamma(\gamma)\} = \int_0^\infty f_\gamma(\gamma)\gamma^{s-1}d\gamma = \kappa\lambda^{-s}\Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right)$ [6, Eq. (2.8)].

Capacity Under Channel Inversion with Fixed Rate (CIFR): In CIFR, the normalized capacity (in bits/sec/Hz) is given by [3, Eq. (46)]

$$C_{\text{CIFR}} = \log_2\left(1 + \left(\int_0^\infty f_\gamma(\gamma)/\gamma d\gamma\right)^{-1}\right). \tag{3}$$

Replacing $f_\gamma(\gamma)$ with its expression in (1) and using the H-function properties in [6, Eq. (1.58)] and [7, Eq. (8.3.4)], the integration yields

$$\int_0^\infty f_\gamma(\gamma)/\gamma d\gamma = \int_0^\infty \frac{\kappa}{\gamma} H_{q,p}^{n,m}\left(\frac{1}{\lambda\gamma} \middle| \begin{matrix}(1 - b_j, B_j)_q \\ (1 - a_j, A_j)_p\end{matrix}\right)d\gamma$$
$$= \kappa \lim_{s\to 0} \Theta_{q,p}^{n,m}\left(s \middle| \begin{matrix}(1 - b_j, B_j)_q \\ (1 - a_j, A_j)_p\end{matrix}\right), \tag{4}$$

which, in turn, yields the following expression for C_{CIFR}

$$C_{\text{CIFR}} = \log_2\left(1 + \left(\kappa \lim_{s\to 0} \Theta_{q,p}^{n,m}\left(s \middle| \begin{matrix}(1 - b_j, B_j)_q \\ (1 - a_j, A_j)_p\end{matrix}\right)\right)^{-1}\right). \tag{5}$$

Eq. (5) is a unified expression for C_{CIFR} that subsumes all those that have been reported earlier in the literature. It is worth mentioning that the expression is very simple as it includes the ratio of products of Gamma functions. In order to check the validity of this expression, we consider the special case of the Weibull fading with a fading parameter 2α and an average signal-to-noise ratio $\bar{\gamma}$, for which $m = q = 1$, $n = p = 0$, $\kappa = \lambda = \Gamma(1 + \frac{1}{\alpha})/\bar{\gamma}$ and $(b_1, B_1) = (1 - \frac{1}{\alpha}, \frac{1}{\alpha})$ [1, Table II]. Substituting these values in (5), it is straightforward to obtain the following expression for the CIFR capacity $C_{\text{CIFR}} = \log_2\left(1 + \frac{\bar{\gamma}}{\Gamma(1+\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})}\right)$, which is identical to [8, Eq. (15)] thus confirming the validity of (5).

Capacity Under Truncated Channel Inversion with Fixed Rate (TIFR): In TIFR, the normalized capacity is given by [3, Eq. (47)]

$$C_{\text{TIFR}} = \log_2\left(1 + \left(\int_{\gamma_0}^\infty f_\gamma(\gamma)/\gamma d\gamma\right)^{-1}\right)(1 - P_{\text{out}}) \tag{6}$$

where $P_{\text{out}} = \text{P}(\gamma \le \gamma_0)$ is the outage probability. The threshold γ_0 is usually chosen numerically such that the channel capacity is maximized. Now, P_{out} can be obtained as

$$P_{\text{out}} = \int_0^{\gamma_0} f_\gamma(\gamma)d\gamma = \frac{\kappa}{2\pi i}\int_{\mathcal{L}} G(1 - s)\lambda^{-s}\Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right)ds$$
$$= \frac{\kappa\gamma_0}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(1 - s)}{\Gamma(2 - s)}(\lambda\gamma_0)^{-s}\Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix}(a_j, A_j)_p \\ (b_j, B_j)_q\end{matrix}\right)ds$$
$$= \frac{\kappa}{\lambda} H_{p+1,q+1}^{m,n+1}\left(\lambda\gamma_0 \middle| \begin{matrix}(1, 1), (a_j + A_j, A_j)_p \\ (b_j + B_j, B_j)_q, (0, 1)\end{matrix}\right). \tag{7}$$

where $G(s) = \mathcal{M}\{u(\gamma) - u(\gamma - \gamma_0)\} = \frac{\gamma_0^s}{s}$ and $u(\gamma)$ is the unit-step function. In the above equation, we started with the definition in (1), used Parseval's relation [7, Eq. (8.3.23)] with some manipulations and, finally, the property in [6, Eq. (1.60)] to get the result in (7). Exact similar steps can be followed to obtain the integration inside the logarithmic function as

$$\int_{\gamma_0}^\infty f_\gamma(\gamma)/\gamma d\gamma = \kappa H_{p+1,q+1}^{m+1,n}\left(\lambda\gamma_0 \middle| \begin{matrix}(a_j, A_j)_p, (1, 1) \\ (0, 1), (b_j, B_j)_q\end{matrix}\right). \tag{8}$$

Combining (7) and (8) in (6), the TIFR capacity expression immediately follows. In order to check the validity of the result, we consider the special case of Rayleigh fading, which can be obtained by setting $m = q = 1$, $n = p = 0$, $\kappa = \lambda = \frac{1}{\bar{\gamma}}$ and $(b_1, B_1) = (0, 1)$ [1, Table II]. Doing so, (7) becomes

$$P_{\text{out}} = H_{1,2}^{1,1}\left(\frac{\gamma_0}{\bar{\gamma}} \middle| \begin{matrix}(1, 1) \\ (1, 1), (0, 1)\end{matrix}\right) = \gamma_{\text{inc}}\left(1, \frac{\gamma_0}{\bar{\gamma}}\right) = e^{-\frac{\gamma_0}{\bar{\gamma}}} \tag{9}$$

where the second equality results from [9, Eq. (4.22)] and $\gamma_{\text{inc}}(\cdot, \cdot)$ is the lower incomplete Gamma function. The final equality results directly from the definition of $\gamma_{\text{inc}}(\cdot, \cdot)$. Also, (8) becomes

$$\int_{\gamma_0}^\infty \frac{f_\gamma(\gamma)}{\gamma}d\gamma = \frac{1}{\bar{\gamma}} H_{1,2}^{2,0}\left(\frac{\gamma_0}{\bar{\gamma}} \middle| \begin{matrix}(1, 1) \\ (0, 1), (0, 1)\end{matrix}\right) = \frac{1}{\bar{\gamma}}E_1\left(\frac{\gamma_0}{\bar{\gamma}}\right) \tag{10}$$

where $E_1(\cdot)$ is the exponential integral of first order. The above result is obtained by using [6, Eq. (1.112)] followed by [10, Eq. (06.06.26.0005.01)]. Combining (9) and (10), the TIFR capacity for Rayleigh fading is obtained, which is identical to [3, Eq. (48)] again confirming the validity of our analysis.

Capacity Under Optimum Rate Adaptation with Fixed Power (ORA): In ORA, the normalized capacity is given by [3, Eq. (29)]

$$C_{\text{ORA}} = \int_0^\infty \log_2(1 + \gamma)f_\gamma(\gamma)d\gamma. \tag{11}$$

It is clear that ORA is effectively the classical Shannon's capacity of the fading channel. An expression for this assuming the H-function fading has already been obtained in [5, Eq. (18)]. Hereafter, we provide a much simpler proof to the same expression. We start by substituting from (1) in (11) and again using Parseval's relation [7, Eq. (8.3.23)] to arrive at

$$C_{\text{ORA}} = \frac{\kappa}{2\pi i \ln 2} \int_{\mathcal{L}} G(1-s)\lambda^{-s} \Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix} (a_j, A_j)_p \\ (b_j, B_j)_q \end{matrix}\right) ds \qquad (12)$$

where $G(s) = \mathcal{M}\{\ln(1+\gamma)\}$. Using the identity $\ln(1+\gamma) = H_{2,2}^{1,2}\left(\gamma \middle| \begin{matrix} (1,1),(1,1) \\ (1,1),(0,1) \end{matrix}\right)$ [9, Case 3.4.3] and the Mellin transform of the H-function mentioned earlier, one gets

$$C_{\text{ORA}} = \frac{\kappa}{2\pi i \ln 2} \int_{\mathcal{L}} \lambda^{-s} \Theta_{2,2}^{1,2}\left(1-s \middle| \begin{matrix} (1,1),(1,1) \\ (1,1),(0,1) \end{matrix}\right) \Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix} (a_j, A_j)_p \\ (b_j, B_j)_q \end{matrix}\right) ds$$

$$= \frac{\kappa}{\lambda \ln 2} H_{p+2,q+2}^{m+2,n+1}\left(\lambda \middle| \begin{matrix} (0,1),(a_j+A_j, A_j)_p,(1,1) \\ (0,1),(0,1),(b_j+B_j, B_j)_q \end{matrix}\right) \qquad (13)$$

where the last line is obtained by using the definition of the Theta function in (2) followed by the H-function property in [6, Eq. (1.60)]. Equation (13) is indeed the same one reported earlier in [5, Eq. (18)].

Capacity Under Optimum Power and Rate Adaptation (OPRA): In OPRA, the normalized capacity is given by [3, Eq. (7)]

$$C_{\text{OPRA}} = \int_0^\infty \log_2(\gamma/\gamma_0)\, u(\gamma-\gamma_0) f_\gamma(\gamma) d\gamma \qquad (14)$$

where γ_0 is a threshold value that must satisfy the following condition

$$\int_{\gamma_0}^\infty (1/\gamma_0 - 1/\gamma)\, f_\gamma(\gamma) d\gamma = 1. \qquad (15)$$

For H-function fading, using Parseval's relation [7, Eq. (8.3.23)], (14) yields an expression, which is identical to (12) but with $G(s) = \mathcal{M}\{\ln(\gamma/\gamma_0)\, u(\gamma-\gamma_0)\}$. Carrying out this Mellin transform is straightforward and the capacity expression becomes

$$C_{\text{OPRA}} = \frac{\kappa}{2\pi i \ln 2} \int_{\mathcal{L}} \frac{\gamma_0^{1-s}}{(1-s)^2} \lambda^{-s} \Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix} (a_j, A_j)_p \\ (b_j, B_j)_q \end{matrix}\right) ds$$

$$= \frac{\kappa\gamma_0}{2\pi i \ln 2} \int_{\mathcal{L}} \left[\frac{\Gamma(s-1)}{\Gamma(s)}\right]^2 (\gamma_0\lambda)^{-s}\, \Theta_{p,q}^{m,n}\left(s \middle| \begin{matrix} (a_j, A_j)_p \\ (b_j, B_j)_q \end{matrix}\right) ds. \qquad (16)$$

Merging the Gamma function terms with those inside the Theta function and using the definition in (2) yields the following H-function result

$$C_{\text{OPRA}} = \frac{\kappa}{\lambda \ln 2} H_{p+2,q+2}^{m+2,n}\left(\lambda\gamma_0 \middle| \begin{matrix} (a_j+A_j, A_j)_p,(1,1),(1,1) \\ (0,1),(0,1),(b_j+B_j, B_j)_q \end{matrix}\right). \qquad (17)$$

Following the exact similar procedure, the condition for γ_0 is found as

$$\kappa H_{p+1,q+1}^{m+1,n}\left(\lambda\gamma_0 \middle| \begin{matrix} (a_j, A_j)_p,(1,1) \\ (-1,1),(b_j, B_j)_q \end{matrix}\right) = 1, \qquad (18)$$

which can be solved numerically. As a special case, we again study the Rayleigh channel via substituting with the values of the parameters mentioned earlier and using the H-function property given by [6, Eq. (1.57)] and similar steps that led to (10) to get

$$C_{\text{OPRA}} = \frac{1}{\ln 2} H_{1,2}^{2,0}\left(\frac{\gamma_0}{\bar{\gamma}} \middle| \begin{matrix} (1,1) \\ (0,1),(0,1) \end{matrix}\right) = \frac{1}{\ln 2} E_1\left(\frac{\gamma_0}{\bar{\gamma}}\right), \qquad (19)$$

which is the same result in [3, Eq. (16)]. A similar procedure can be carried out to prove the correctness of the condition in (18) but is not shown here due to space limitations.

Numerical Results: Using the expressions obtained above, it is possible to evaluate the channel capacity over a variety of simple and composite fading distributions such as Rayleigh-exponential, Weibull-gamma, K-distribution and generalised-K among many others. We choose two generalised channel models to report results for, namely, the extended generalised Gamma [1, Table V] and the generalised-K [1, Table IV]. Figs. 1 and 2 show the normalized TIFR and OPRA capacity, respectively, for the two channels for two cases each. We chose those capacities as representative examples. For the extended generalised Gamma, the values of the channel parameters for case 1 are $(m_\ell, m_{s\ell}, \xi_\ell, \xi_{s\ell}) = (1,5,1,2)$, while for case 2, we set $(m_\ell, m_{s\ell}, \xi_\ell, \xi_{s\ell}) = (1,8,2,0.8)$. As for the generalised-K, in case 1, we have $(m_{s\ell}, m_s) = (1.5, 6)$, while in case 2, we choose $(m_{s\ell}, m_s) = (3,4)$. Clearly, the expressions are very versatile and can present results for a variety of channel models in a straightforward and easy manner.

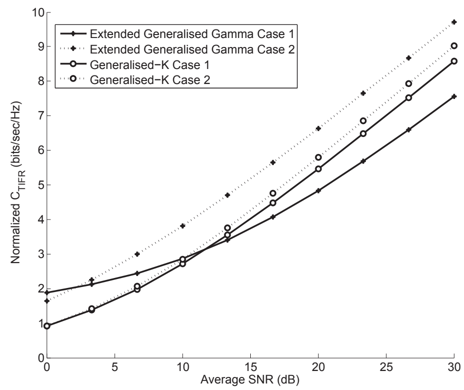

Fig. 1 *TIFR capacity for the extended generalised Gamma and the generalised-K channels for two different cases.*

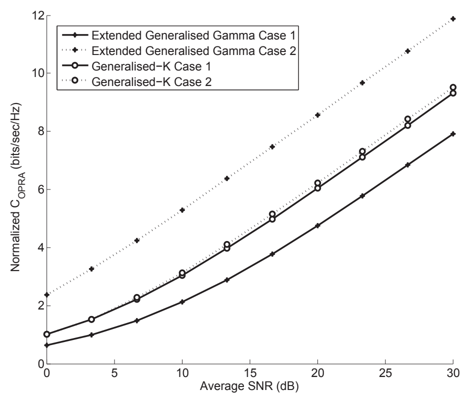

Fig. 2 *OPRA capacity for the extended generalised Gamma and the generalised-K channels for two different cases.*

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