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Advances in periodic difference equations with open problems

Ziyad AlSharawi†, Jose Cánovas‡,* Antonio Linero§
†Department of Mathematics and Statistics, Sultan Qaboos University,
P. O. Box 36, PC 123, Al-Khod, Sultanate of Oman
Email: alsha1zm@alsharawi.info

‡Departamento de Matemática Aplicada y Estadística,
Universidad Politécnica de Cartagena,
Paseo de Alfonso XIII 30203, Cartagena, Murcia, Spain
Email: jose.canovas@upct.es

§Departamento de Matemáticas, Universidad de Murcia,
Campus de Espinardo, Murcia 30100, Spain
Email: lineroba@um.es

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Abstract

In this paper, we review some recent results on the dynamics of semi-dynamical systems generated by the iteration of a periodic sequence of continuous maps. In particular, we state several open problems focused on the structure of periodic orbits, forcing between periodic orbits, sharing periodic orbits, folding and unfolding periodic systems, and on applications of periodic systems.

1 Introduction

Let $C(I)$ denote the set of continuous maps $f : I \to I$ where I is a compact subinterval of the real line. We consider $f_0, ..., f_{p-1} \in C(I)$. These maps generate a semi-dynamical system [34], which we denote by $(I, [f_0, ..., f_{p-1}])$. For any $x \in I$, the

*Corresponding author: jose.canovas@upct.es

orbit through x is denoted by $\mathrm{Orb}(x, [f_0, ..., f_{p-1}])$ and given by the solution of the non–autonomous difference equation

$$\begin{cases} x_0 = x, \\ x_{n+1} = f_{n \bmod p}(x_n). \end{cases} \tag{1.1}$$

The number p is called the period of the system and it is always considered to be minimal.

Periodic difference equations have been studied by several authors recently (see for instance [6, 8, 9, 11, 15, 16, 24, 29, 30, 31, 34]. The interest for studying periodic discrete systems is motivated by applications in population dynamics (see e.g. [28] and [40]) and economic dynamics (see the so-called duopolies (see e.g. [46] and [49])). When one observes the orbit (x_n) of a point $x \in I$, it is easy to note that the subsequences (x_{pn}), (x_{pn+1}),...,(x_{pn+p-1}) are respectively the orbits of the initial points

$$x_0, \quad f_0(x_0), \quad ... \quad , (f_{p-2} \circ ... \circ f_0)(x_0)$$

under the iteration of the individual maps

$$F_0, \quad F_1, \quad F_2, \quad \ldots \quad , F_{p-1},$$

where $F_j = f_{(p-1+j) \bmod p} \circ ... \circ f_{j+1} \circ f_j$ for $j = 0, 1, \ldots p-1$. Therefore, one might expect that the dynamics can be completely given by the dynamics of the above individual maps. Indeed, this is true for some characteristics of the dynamical system. For instance, the topological entropy of $[f_0, ..., f_{p-1}]$ can be computed by means of the topological entropy of $f_{p-1} \circ ... \circ f_0$ (see [42]). On the other hand, the ω–limit set $\omega(x, [f_0, ..., f_{p-1}])$, which is the set of limit points of the orbit with initial condition x, can be obtained from the equality

$$\begin{aligned} \omega(x, [f_0, ..., f_p]) &= \omega(x, f_{p-1} \circ ... \circ f_0) \cup \omega(f_0(x), f_0 \circ f_{p-1} \circ ... \circ f_1) \cup \ldots \\ \ldots &\cup \omega((f_{p-2} \circ ... \circ f_0)(x), f_{p-2} \circ ... \circ f_0 \circ f_{p-1}) \\ &= \omega(x, F_0) \cup \omega(f_0(x), F_1) \cup \ldots \cup \omega((f_{p-2} \circ ... \circ f_0)(x), F_{p-1}) \end{aligned}$$

where each $\omega(z_j, F_j)$ is meant the set of limit points of the orbit of $z_j = (f_{j-1} \circ ... \circ f_0)(x)$ under the interval map F_j, $j = 0, 1, \ldots, p-1$, (here, $z_0 = x$).

The aim of this paper is to show that, even when many dynamical properties can be studied by the folded dynamical systems, there are several open problems that deserve investigation. The paper is organized in sections and each section covers a topic that includes some proposed open problems. In Section 2 we deal with the set of periods $\mathrm{Per}[f_0, \ldots, f_{p-1}]$ of periodic non-autonomous systems. In Section 3 we analyze how this set can be altered by the effect of folding some maps of the periodic non-autonomous system in order to obtain a new system, of possibly shorter period. After this, we present in Section 4, the question of studying the resulting period when we combine strings of two given periodic sequences. Another problem related with periodic orbits appears in Section 5: it is an open problem to determine whether or not the intersection of the sets of periodic points of two commuting interval maps is empty. Section 6 is devoted to the Parrondo's paradox. Finally, we present some interesting applications related to the dynamics of population models described by periodic non–autonomous systems.

2 Periodic orbits in periodic non–autonomous systems

When Eq. (1.1) is composed of one map, say f, an orbit $\mathrm{Orb}(x,f)=(x_n)$ is said to be periodic if there is $q\in\mathbb{N}:=\{1,2,\dots\}$ such that $x_{n+q}=x_n$ for all $n\geq 0$. The smallest number q satisfying this condition is called the period or order, and denoted by $\mathrm{ord}_f(x)$. In the case of discrete dynamical systems on the interval I, the well–known Sharkovsky's theorem characterizes the set of periods of f, denoted $\mathrm{Per}(f)$. More precisely, we consider the following order in the set of natural numbers $\mathbb{N}$.

$$3 \succ_{\mathrm{s}} 5 \succ_{\mathrm{s}} 7 \succ_{\mathrm{s}} \dots \succ_{\mathrm{s}} 2\cdot 3 \succ_{\mathrm{s}} 2\cdot 5 \succ_{\mathrm{s}} 2\cdot 7 \succ_{\mathrm{s}} \dots$$
$$2^n\cdot 3 \succ_{\mathrm{s}} 2^n\cdot 5 \succ_{\mathrm{s}} 2^n\cdot 7 \succ_{\mathrm{s}} \dots \succ_{\mathrm{s}} 2^{n+1} \succ_{\mathrm{s}} 2^n \succ_{\mathrm{s}} \dots \succ_{\mathrm{s}} 2 \succ_{\mathrm{s}} 1.$$

For $n\in\mathbb{N}\cup\{2^\infty\}$, define $\mathcal{S}(n)=\{m\in\mathbb{N}: n>_{\mathrm{s}} m\}\cup\{n\}$ and $\mathcal{S}(2^\infty)=\{2^n: n\in\mathbb{N}\cup\{0\}\}$. Sharkovsky's theorem states that if f has a periodic orbit (periodic sequence) of period n, then it has periodic points (periodic sequences) of period $m\in S(n)$. Moreover, for any $n\in\mathbb{N}\cup\{2^\infty\}$ there is $f\in C(I)$ such that $\mathrm{Per}(f)=\mathcal{S}(n)$ (see [54] or [33] for a recent proof of Sharkovsky's theorem).

In the case of periodic non-autonomous systems, a generalization of Sharkovsky's theorem is given in [11]. For a fixed $p\in\mathbb{N}$, consider a p-periodic system defined by the maps $[f_0,\dots,f_{p-1}]$. For $q\in\mathbb{N}$, define the clusters

$$\mathcal{A}_{p,q}=\{n:\mathrm{lcm}(n,p)=q\cdot p\}=\{n: q=\frac{n}{\gcd(n,p)}\}.$$

Notice that $p\cdot q\in\mathcal{A}_{p,q}$. Now, define the equivalence relation "$\sim_p$" on $\mathbb{N}$ by stating that $n\sim_p m$, $n,m\in\mathbb{N}$, if and only if n and m belong to the same set $\mathcal{A}_{p,q}$ for some $q\in\mathbb{N}$. If we denote any equivalence class $\mathcal{A}_{p,q}$ by $[q]$, we define the order on $\mathbb{N}/\sim_p$ by $[n]>_{\mathrm{s}}[m]$ if and only if $n>_{\mathrm{s}} m$. Now, $[q]\in\mathrm{Per}([f_0,\dots,f_{p-1}])/\sim_p$ denotes that $\mathcal{A}_{p,q}\cap\mathrm{Per}([f_0,\dots,f_{p-1}])$ is nonempty. The generalization in [11] shows if $[n]\in\mathrm{Per}([f_0,\dots,f_{p-1}])/\sim_p$, then for any $[m]\in\mathbb{N}/\sim_p$ such that $[n]>_{\mathrm{s}}[m]$, we have $[m]\in\mathrm{Per}([f_0,\dots,f_{p-1}])/\sim_p$. The proof of this result is based on two facts:

Sharkovsky's theorem and the fact that if $m \in \mathrm{Per}([f_0, ..., f_{p-1}])$ and $m \in [q]$, then $q \in \mathrm{Per}(f_{p-1} \circ ... \circ f_0)$.

When $p = 2$, another approach was used in [24] to characterize the structure of the set of periods $\mathrm{Per}([f_0, f_1])$. More precisely, set

$$\mathbb{N}^* := \mathbb{N} \setminus (\{2n-1 : n \in \mathbb{N}\} \cup \{2\}).$$

The following result is given in [24].

Theorem 2.1. *Each of the following holds true for a 2-periodic system:*

(a) *If $[f_0, f_1]$ has a periodic orbit of period $n \in \mathbb{N}^* \cup \{2^\infty\}$, then $\mathcal{S}(n)\backslash\{1,2\} \subset \mathrm{Per}[f_0, f_1]$.*

(b) *If $2n+1 \in \mathrm{Per}[f_0, f_1]$, $n \geq 1$, then $\mathcal{S}(2 \cdot 3)\backslash\{1\} \subset \mathrm{Per}[f_0, f_1]$.*

(c) *There is a 2-periodic system $[f_0, f_1]$ such that $\mathrm{Per}([f_0, f_1])$ is $\{1\}$, $\{2\}$ or $\{1,2\}$.*

(d) *For any $n \in \mathbb{N}^* \cup \{2^\infty\}$: There is a 2-periodic system $[f_0, f_1]$ such that one of the following is satisfied.*

d.1. $\mathrm{Per}([f_0, f_1]) = \mathcal{S}(n)$.

d.2. $\mathrm{Per}([f_0, f_1]) = \mathcal{S}(n)\backslash\{1\}$.

d.3. $\mathrm{Per}([f_0, f_1]) = \mathcal{S}(n)\backslash\{2\}$.

(e) *For any subset of odd numbers $\mathrm{Imp} \subseteq \{2n+1 : n \in \mathbb{N}\}$ there is a 2-periodic system $[f_0, f_1]$ such that one of the following is satisfied.*

e.1. $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup (\mathcal{S}(2 \cdot 3)\backslash\{1\})$.

e.2. $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup \mathcal{S}(2 \cdot 3)$.

Notice that the case $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup (\mathcal{S}(2 \cdot 3)\backslash\{2\})$ is not allowed, that is, if $2n+1 \in \mathrm{Per}([f_0, f_1])$ for some $n \in \mathbb{N}$, then automatically $2 \in \mathrm{Per}([f_0, f_1])$. In addition, for $n \in \mathbb{N}^* \cup \{2^\infty\}$, $n \neq 2 \cdot 3$, there are no continuous maps $f_0, f_1 \in C(I)$ such that $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup (\mathcal{S}(n)\backslash\{1\})$ or $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup (\mathcal{S}(n)\backslash\{2\})$ or $\mathrm{Per}([f_0, f_1]) = \mathrm{Imp} \cup \mathcal{S}(n)$.
The following frame summarizes the forcing (where $n >_2 m$ is meant that the presence of a period n in the alternated system forces the existence of periodic sequences having order m):

$$\{2 \cdot n + 1 : n \in \mathbb{N}\} >_2 2 \cdot 3 >_2 2 \cdot 5 >_2 2 \cdot 7 >_2 ...$$
$$2^n \cdot 3 >_2 2^n \cdot 5 >_2 2^n \cdot 7 >_2 ... >_2 2^n >_2 ... >_2 2^2 >_2 (1 \text{ or/and } 2).$$

The generalization of Sharkovsky's theorem given by AlSharawi et al. in [11] lacks the details about the forcing relationship within each equivalence class $[q] = \mathcal{A}_{p,q}$. On the other hand, the result of Cánovas and Linero in [24] gives the exact forcing between cycles when $p = 2$, but lacks the generality for $p > 2$. Since the results in [11] and [22], several attempts have been made to give the exact forcing within each equivalence class $[q]$ [6, 8, 15]. Although progress has been made in special cases, the general case is still open, which motivates our first open problem.

Open Problem 2.1. *Extend Theorem 2.1 to periodic sequences of maps of arbitrary period, i.e., characterize the set of periods* $\mathrm{Per}([f_0, ..., f_{p-1}])$ *for any positive integer* p.

In each equivalence class or cluster $[q] = \mathcal{A}_{p,q}$, there is one period, namely pq, that does not depend on the intersection between the maps $f_0, f_1, \ldots, f_{p-1}$, while the other periods need certain intersections between the maps [8, 15]. This observation leads to dividing the periods into the ones that are generic properties of the intersections, and the ones that are generic properties of the iterations. Indeed, establishing a connection between those two sets was one of the objectives in [8, 6, 12]. Examples were constructed [6] to show that the existence of $m \in \mathrm{Per}([f_0, ..., f_{p-1}])$ for some $m \in \mathcal{A}_{p,q}$, where $q > 1$ is a power of 2, does not guarantee that $pq \in \mathrm{Per}([f_0, ..., f_{p-1}])$. However, the following problem is still open.

Open Problem 2.2. *Suppose* $m \in \mathrm{Per}([f_0, ..., f_{p-1}])$ *for some* $m \in \mathcal{A}_{p,q}$*, where* $q > 1$ *is an odd number. Prove that* $pq \in \mathrm{Per}([f_0, ..., f_{p-1}])$.

It is well known that Sharkovsky's theorem works under certain modifications for other one-dimensional spaces, as the circle -with some modifications as a consequence of the degree of a circle map-, even for classes of n-dimensional continuous maps, for instance the so-called triangular maps $G(x_1, x_2, \ldots, x_n) = (g_1(x_1), g_2(x_1, x_2), \ldots, g_n(x_1, x_2, \ldots, x_n))$ (see [5] for more details). To this respect:

Open Problem 2.3. *Extend Theorem 2.1 to periodic sequences of two continuous circle maps, that is, characterize the set of periods* $\mathrm{Per}([f_0, f_1])$*, where* $f_0, f_1 : \mathbb{S}^1 \to \mathbb{S}^1$ *are continuous.*

3 Folding in periodic systems

For a p-periodic system $[f_0, ..., f_{p-1}]$, it is possible to fold some of the maps to obtain a system of possibly shorter period. For instance, an obvious situation is the p-fold map $F_0 := f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$, which changes the p-periodic non-autonomous system into an autonomous system. In general, for any $1 \leq k \leq p - 1$, we can fold the maps

$$f_{k-1} \circ \cdots \circ f_0 =: F_0,\ f_{2k-1} \circ \cdots \circ f_k =: F_1,\ f_{3k-1} \circ \cdots \circ f_{2k} =: F_2, \cdots \tag{3.1}$$

to form another periodic system $[F_0, F_1, ..., F_{\frac{p}{\gcd(p,k)}}]$. Some caution must be made here about the period of the new periodic system. It may not be $\frac{p}{\gcd(p,k)}$. Indeed, take the 4-periodic system $[f, f^{-1}, f^{-1}, f]$ and $k = 2$, then the folded system $[F_0, F_1]$ is 1-periodic rather than 2-periodic. The notion of folding was introduced in [6]. The scenario of having $[F_0, F_1, ..., F_{\frac{p}{\gcd(p,k)}}]$ with a period less than $\frac{p}{\gcd(p,k)}$ is called degenerate scenario and has been avoided. The case when k is a divisor of p is studied in [12] and connections between the cycles of the folded and unfolded systems have been established. However, for general k, the connection between the folded and unfolded systems still open for further investigation, which motives the next open problem.

Open Problem 3.1. *Consider the p-periodic system* $[f_0, ..., f_{p-1}]$*, and let* $1 \leq k \leq p$*. Consider the maps* $F_0, F_1, \ldots, F_{\frac{p}{\gcd(p,k)}}$ *as defined in (3.1). What is the relationship between* $Per([f_0, ..., f_{p-1}])$ *and* $Per([F_0, F_1, ..., F_{\frac{p}{\gcd(p,k)}}])$*?*

Periodic difference equations with delay of the form $y_{n+1} = g_{n \bmod p}(y_{n-(k-1)})$ have been studied by AlSharawi et al. in [10], and a characterization of the periodic structures were given. To visualize the orbits in this case, consider $p = 6$ and $k = 4$, then orbits of $y_{n+1} = g_{n \bmod 6}(y_{n-3})$ can be written in matrix form as

$$\begin{array}{cccc} y_{-3} & y_{-2} & y_{-1} & y_0 \\ g_0 & g_1 & g_2 & g_3 \\ g_4 & g_5 & g_0 & g_1 \\ g_2 & g_3 & g_4 & g_5. \end{array}$$

From the columns of this matrix, observe that each column gives a periodic system of period $\frac{p}{\gcd(p,k)}$, which can be assumed to be the minimal period. This observation motivates the following problem.

Open Problem 3.2. *Suppose there is a p-periodic system* $[f_0, ..., f_{p-1}]$*, which is unknown to us, but we know one of its folded systems* $[F_0, F_1, \ldots, F_{\frac{p}{\gcd(p,k)}}]$*. What kind of similarity (if any) in periodic structure do we have between the unfolded p-periodic system* $[f_0, ..., f_{p-1}]$ *and the periodic system with delay* $x_{n+1} = F_n(x_{n-(k-1)})$*?*

4 Merging periodic sequences

In the unfolding process of a $\frac{p}{k}-$periodic system $[F_0, F_1, \ldots, F_{\frac{p}{k}}]$, we find ourselves dealing with sequences that are merged in a certain way [12]. Therefore, we find the notion of merging two periodic sequences to be related to the topic of the previous section, and therefore, we find it is worth addressing here. Suppose we have two periodic sequences $\{a_n\}$ and $\{b_n\}$ of periods q_1 and q_2, respectively. The two sequences can be thought of as two periodic signals or codes coming out of two machines. After each string of length k_1 produced by the first machine (a k_1 string of $\{a_n\}$), the second machine releases a k_2 string (a k_2 string of $\{b_n\}$). The obtained signal has the structure $[a_n, b_n] :=$

$$\overbrace{a_1, a_2, \ldots, a_{k_1}}^{k_1 \text{ string}}, \overbrace{b_1, b_2, \ldots, b_{k_2}}^{k_2 \text{ string}}, \overbrace{a_{k_1+1}, a_{k_1+2}, \ldots, a_{2k_1}}^{k_1 \text{ string}}, b_{k_2+1}, \ldots \tag{4.1}$$

Before we proceed, we clarify the notion by an illustrative example.

Example 4.1. *For* $n \in \mathbb{N}$, *consider* $a_n = n \bmod 4$ *and* $b_n = 4 + (n \bmod 6)$. *Thus,* $\{a_n\}$ *is periodic of period* $q_1 := 4$ *and* $\{b_n\}$ *is periodic of period* $q_2 := 6$.

(i) If $k_1 = 2$ *and* $k_2 = 3$, *then*

$$[a_n, b_n] = \{0, 1, 4, 5, 6, 2, 3, 7, 8, 9, 0, 1, 4, 5, 6, \cdots\},$$

and the period of the formed sequence is 10.

(ii) If, $k_1 = 2$ *and* $k_2 = 4$, *then*

$$\begin{aligned}[a_n, b_n] &= \{0, 1, 4, 5, 6, 7, 2, 3, 8, 9, 4, 5, 0, 1, 6, 7, 8, 9, \\ &\quad 2, 3, 4, 5, 6, 7, 0, 1, 8, 9, 4, 5, 2, 3, 6, 7, 8, 9, 0, 1 \cdots\}\end{aligned}$$

and the period of the formed sequence is 36.

For a better understanding of the structure of the formed sequence $[a_n, b_n]$, we write its elements in matrix form as follows:

$$\begin{array}{cccccccc} a_0 & a_1 & \cdots & a_{k_1-1} & b_0 & b_1 & \cdots & b_{k_2-1} \\ a_{k_1} & a_{k_1+1} & \cdots & a_{2k_1-1} & b_{k_2} & b_{k_2+1} & \cdots & b_{2k_2-1} \\ a_{2k_1} & a_{2k_1+1} & \cdots & a_{3k_1-1} & b_{2k_2} & b_{2k_2+1} & \cdots & b_{3k_2-1} \\ a_{3k_1} & a_{3k_1+1} & \cdots & a_{4k_1-1} & b_{3k_2} & b_{3k_2+1} & \cdots & b_{4k_2-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \ddots \end{array} \tag{4.2}$$

Now, reading this matrix row by row from left to right gives the formed sequence $[a_n, b_n]$. It is obvious that we obtain a periodic sequence in each column. In some cases, it is easy to deduce the new period of the sequence obtained from the merging process, for instance when the merged sequences are disjoint (we establish here the result without proof):

Lemma 4.1. *For* $n \in \mathbb{N}$, *let* $\{a_n\}$ *and* $\{b_n\}$ *be two disjoint periodic sequences of periods* q_1 *and* q_2, *respectively. Also, let* $1 \le k_1 \le q_1$ *and* $1 \le k_2 \le q_2$. *The period of the sequence* $[a_n, b_n]$ *formed in (4.1) is of minimal period* kq, *where* $k = k_1 + k_2$ *and* $q = \operatorname{lcm}\left(\frac{q_1}{\gcd(k_1, q_1)}, \frac{q_2}{\gcd(k_2, q_2)}\right)$.

The condition to have the two sequences disjoint is a luxury that one may not have, which leads us to state the following general problem:

Open Problem 4.1. *For* $n \in \mathbb{N}$, *let* $\{a_n\}$ *and* $\{b_n\}$ *be two periodic sequences of periods* q_1 *and* q_2, *respectively. Find the minimal period of the sequence* $[a_n, b_n]$ *formed in (4.1).*

5 Commuting maps and the problem of sharing periodic orbits

Denote by $\mathrm{Fix}(f)$ and $\mathrm{P}(f)$ the set of fixed and periodic points of a map $f \in C(I)$, respectively. We consider two maps $f_0, f_1 \in C(I)$ such that they commute, that is, $f_0 \circ f_1 = f_1 \circ f_0$.

In the fifties of the 20-th century, some authors posed independently the problem of proving whether two commuting continuous interval maps share fixed points. The problem is answered in affirmative for polynomials, as J. F. Ritt pointed out in [51]. Other cases, under restrictive conditions, have also a positive answer (for instance, we mention [32] or [55]). The question on whether $\mathrm{Fix}(f_0) \cap \mathrm{Fix}(f_1)$ is nonempty, that is, f_0 and f_1 have a common fixed point, was open for a long time [39]. Finally, Boyce ([21]) and Huneke ([38]) found simultaneously counterexamples which show that in general the answer is negative, there exist two continuous commuting interval maps which do not share any fixed point. The counterexamples constructed in [21] and [38] are given by two maps f_0 and f_1 that share periodic points.

Since then, the research on this topic was concentrated in several directions. For instance, to extend the problem to other compact metric spaces or to particular classes of continuous maps ([41], [35], [43]). The problem has been also posed in terms of sharing periodic points which are not necessarily fixed points (see [3] and [57]). Then, it can be expected to raise the following question:

Open Problem 5.1. *Is it true that* $\mathrm{P}(f_0) \cap \mathrm{P}(f_1) \neq \emptyset$ *for commuting continuous interval maps* $f_0, f_1 \in C(I)$*?*

Fixed and periodic points are the strongest type of recurrence in dynamical systems. There are weaker notions of recurrence that contain the sets of fixed and periodic points. Namely, a point $x \in X$ is called *recurrent* if for any open neighborhood U of x there is an increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $f^{n_i}(x) \in U$. If the sequence $\{n_i\}_{i=1}^{\infty}$ has bounded gaps, the point is called *uniformly recurrent*. If $n_i = ki$ for some $k \in \mathbb{N}$ the point is called *almost periodic*. Denote by $\mathrm{Rec}(f)$, $\mathrm{UR}(f)$ and $\mathrm{AP}(f)$ the sets of recurrent, uniformly recurrent and almost periodic points. It is clear from the definitions that

$$\mathrm{Fix}(f) \subseteq \mathrm{P}(f) \subseteq \mathrm{AP}(f) \subseteq \mathrm{UR}(f) \subseteq \mathrm{Rec}(f).$$

Following the Sharkovsky's order of natural numbers, let $\mathcal{T}_1 = \{f \in C(I) : \mathrm{P}(f)$ is closed$\}$, $\mathcal{T}_2 = \{f \in C(I) : f$ has periodic points of period 2^n, $n \geq 0\}$ and $\mathcal{T}_3 = \{f \in C(I) : f$ has a periodic point which is not a power of two$\}$. The next result was proved in [23].

Theorem 5.1. *Assume* $f_0, f_1 \in C(I)$ *commute. Then* *(a) If* $f_0 \in \mathcal{T}_1$*, then* $\mathrm{Fix}(f_0) \cap \mathrm{P}(f_1) \neq \emptyset$*.*

(b) If $f_0 \in \mathcal{T}_2$*, then* $\mathrm{Fix}(f_0) \cap \mathrm{AP}(f_1) \neq \emptyset$*.*

(c) If $f_0 \in \mathcal{T}_3$*, then* $\mathrm{Fix}(f_0) \cap \mathrm{UR}(f_1) \neq \emptyset$*.*

The above result proves Open Problem 5.1 for maps which are simple from the point of view of dynamics. Note that maps of type $\mathcal{T}_3$ have positive topological entropy, and therefore, they are chaotic in the sense of Li and Yorke. Chaotic maps in the sense of Li and Yorke may also exist in the family $\mathcal{T}_2$, but they cannot be found in $\mathcal{T}_1$, which contains the set of continuous interval maps with finite set of periods (see [18] or [23]). So, if one has to look for counterexamples for Open Problem 5.1, he/she should construct maps having both infinitely many periodic points. In addition, they cannot have a finite number of monotonicity pieces (see [21] and [38]).

We finish this section by a problem that links Theorem 2.1 and Open problem 5.1.

Open Problem 5.2. *Find* $\mathrm{Per}([f_0, f_1])$ *for commuting maps* $f_0, f_1 \in C(I)$*. More precisely, can the examples of Theorem 2.1 be constructed such that* f_0 *and* f_1 *commute?*

6 The Parrondo's paradox

Parrondo's paradox [36] has become an active area of research in many applied sciences like Physics [48], Economy [58] or Biomathematics [59]. As a first approach, we can say that it appears when we alternate different games in a stochastic or deterministic way. Parrondo's paradox exists when the behavior of individual systems and the combined one are completely different. For discrete dynamical systems, the paradox was formulated in [4] by showing that the phenomenon "chaos+chaos=order" and "order+order=chaos" are possible when considering periodic combinations of 1-dimensional quadratic maps. Similar results have been obtained by Boyarsky and collaborators in the random combination of piecewise smooth maps [20]. On the other hand, it was shown in [22] that in some particular cases the paradox is not possible.

The dynamic Parrondo's paradox was studied in detail in [25] as follows. For a map $f \in C(I)$, $I = [0,1]$, denote by $\mathcal{D}(f)$ the set of dynamic properties of f (for instance to have positive topological entropy or exhibiting chaos in the sense of Li and Yorke), and define $\mathcal{D}([f_0, f_1])$ similarly. Let $J = [a,b] \subseteq I = [0,1]$ and denote by $\varphi_J : J \to I$ a linear map such that $\varphi_J(a) = 0$ and $\varphi_J(b) = 1$. Define $f_J : [0,1] \to [0,1]$ by $f_J(x) = \varphi_J^{-1} \circ f \circ \varphi_J(x)$ if $x \in J$, $f_J(0) = 0$, $f_J(1) = 1$, and linear on any connected component of $[0,1] \setminus J$. A dynamic property $P \in \mathcal{D}(f)$ is an L–property if for any continuous map f and any compact subinterval $J \subseteq [0,1]$, it is held that $P \in \mathcal{D}(f) \cap \mathcal{D}(f_J)$. The fact that if $f_0, f_1 \in C(I)$, then the dynamics of the sequence $[f_0, f_1]$ is complicated (or simple) if and only if the dynamics of $f_0 \circ f_1$

is complicated (or simple), jointly with L–properties are the key for analyzing the Parrondo's paradox as the following result shows [25].

Theorem 6.1. *Let P_i, $i = 1, 2, 3$, be L–properties. Then there are $f_0, f_1 \in C(I)$ such that $P_1 \in \mathcal{D}(f_0)$, $P_2 \in \mathcal{D}(f_1)$ and $P_3 \in \mathcal{D}(f_0 \circ f_1)$.*

In particular, we can construct maps such that f_0 and f_1 have a complicated (simple) L–property and $f_0 \circ f_1$ has not this property. $[P \in \mathcal{D}(f_0) \cap \mathcal{D}(f_1)$ and $P \notin \mathcal{D}(f_0 \circ f_1)]$. For instance, we consider the topological entropy (see e.g. [2] or [5] for definition and basic properties of topological entropy), which is a useful tool to decide whether a map has a complicated dynamics. From Theorem 6.1, we can construct two continuous interval maps, f_0 and f_1, with zero topological entropy (and hence simple) such that $f_0 \circ f_1$ has positive topological entropy (and therefore a complicated dynamics), because the properties zero topological entropy and positive topological entropy are L–properties. However, we must emphasize that, although Theorem 6.1 shows the existence of paradox in a general way for a very large list of dynamical properties, the constructions made for proving it cannot be done when one consider fixed maps like for instance a member of the logistic family $f_a(x) = ax(1-x)$, $1 \leq a \leq 4$ and $x \in [0, 1]$.

Open Problem 6.1. *State an analogous result to Theorem 6.1 when the maps f_0 and f_1 commute.*

Consider the well–known logistic family $f_a(x) = ax(1 - x)$, $a \in [1, 4]$. More precisely, we consider two maps f_a and f_b and wonder about the existence of paradox for parameters a and b. In [27], the existence of Parrondo's paradox is shown for several parameters values. However, although $f_a \circ f_b$ may exhibit the paradox, it is observed that several combinations like $f_a \circ f_b \circ f_a$ do not exhibit the paradox.

Open Problem 6.2. *Characterize the set of parameters $a, b \in [1, 4]$ such that any combination of maps f_a and f_b exhibit the Parrondo's paradox. Is this set nonempty?*

Denote the topological entropy of a continuous map f by $h(f)$. In [26], numerical simulations show that Parrondo's paradox cannot be exhibited by maps with positive topological entropy, that is, if $\min\{h(f_a), h(f_b)\} > 0$, then numerical simulations show that $h(f_a \circ f_b) > 0$, and therefore the Parrondo's paradox cannot be exhibited (see Figure 1).

Open Problem 6.3. *In the logistic family, prove or disprove that* $\min\{h(f_a), h(f_b)\} > 0$ *implies that* $h(f_a \circ f_b) > 0$.

7 Applications

In population dynamics, one dimensional models usually take the form $x_{n+1} = x_n f(x_n)$, where x_n represents the density of a population at discrete time n and $f(t)$

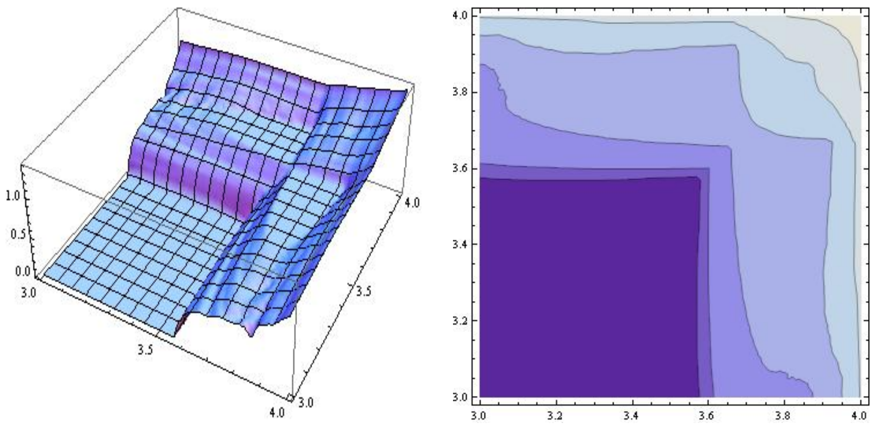

Figure 1: On the left we show the topological entropy of F with accuracy 10^{-3}. On the right, we present the projection of topological entropy where stronger colors means lower topological entropy. The darkest region represents those parameter values for which the topological entropy is zero up to the prescribed accuracy.

is a function that reflects certain characteristics of the studied species. For instance, $f(t) = \frac{k\mu}{k+(\mu-1)t}$, $\mu > 1$ is used for the Beverton-Holt model [17], $f(t) = at(b-t)$ is used for the logistic model [47], and $f(t) = be^{-kt}$ is used for the Ricker model [50]. Forcing periodic harvesting or stocking in a deterministic environment leads to investigating the dynamics of models in the from

$$x_{n+1} = x_n f(x_n) \pm h_n, \tag{7.1}$$

where $\{h_n\}$ is a p-periodic sequence that represents harvesting or stocking quotas. See [7] and the references therein for more details and some open questions. We find Problem 3.1 in [7] to be suitable within the context of this paper.

Open Problem 7.1. *Consider Eq. (7.1) with stocking (i.e. $+h_n$) and assume this equation has a global attractor (like when $f(t) = \frac{bt}{1+t}$ [14]). Let $\{\hat{h}_n\}$ be a permutation of $\{h_n\}$. Define x_{av} and $\hat{x}_{av}$ to be the average of the global attractors associated with $\{h_n\}$ and $\{\hat{h}_n\}$, respectively. How does x_{av} relate to $\hat{x}_{av}$?*

Although it is tempting to believe that increasing constant yield harvesting in population models leads to a decline in the population, recent results show otherwise [56, 52] and the phenomenon is known as the hydra effect [1]. In fact, this notion lead to a fertile area of research; see for instance [53, 44, 45] and the references therein. When we confine models in Eq. (7.1) to contest-competition models and force the harvesting to be constant yield harvesting (i.e., $h_n = h$ for all n), then the hydra effect is not known to take place. For instance, if we take the Beverton-Holt model with harvesting [13], then the global attractor is decreasing in h. However, the effect of periodic harvesting is not characterized yet, which motivates our next problem.

Open Problem 7.2. *Consider Eq. (7.1) with harvesting (i.e. $-h_n$). Investigate the effect of ordering the elements of the sequence $\{h_n\}$ on the population. On other words, assume that we have a periodic sequence of harvesting quotas but we have the freedom to permute its elements. Which permutation of $\{h_n\}$ plays on the advantage of the population in terms of the basin of attraction and in terms of the arithmetic average of the attractor?*

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