Commutants of the sum of two quasihomogeneous Toeplitz operators

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Research Article

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Commutants of the sum of two quasihomogeneous Toeplitz operators

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Abstract: A major open question in the theory of Toeplitz operator on the Bergman space of the unit disk of the complex plane is the complete characterization of the set of all Toeplitz operators that commute with a given operator. Researchers showed that when a sum $S = T_{e^{im\theta}f} + T_{e^{il\theta}g}$, where f and g are radial functions, commutes with a sum $T = T_{e^{ip\theta}r^{(2M+1)p}} + T_{e^{is\theta}r^{(2N+1)s}}$, then S must be of the form $S = cT$, where c is a constant. In this article, we will replace $r^{(2M+1)p}$ and $r^{(2N+1)s}$ with r^n and r^d, where n and d are in $\mathbb{N}$, and we will show that the same result holds.

Keywords: Toeplitz operators, quasihomogeneous symbol, Mellin transform, Gamma function

MSC 2020: Primary 47B35, Secondary 47L80

1 Introduction

In the complex plane $\mathbb{C}$, let $\mathbb{D}$ be the open unit disk, and $\mathrm{d}A = r\mathrm{d}r\frac{\mathrm{d}\theta}{\pi}$ be the normalized Lebesgue measure on $\mathbb{D}$, where (r, θ) are the polar coordinates. The Hilbert space $L^2(\mathbb{D}, \mathrm{d}A)$ consists of all square-integrable functions on $\mathbb{D}$ with respect to the measure $\mathrm{d}A$.

The classical unweighted Bergman space $L_a^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D}, \mathrm{d}A)$ consisting of all analytic functions on D. Moreover, the set $\{z^n : n = 0, 1, 2, \ldots\}$ is an orthogonal basis for $L_a^2(\mathbb{D})$. Since $L_a^2(\mathbb{D})$ is closed, the orthogonal projection P from $L^2(\mathbb{D}, \mathrm{d}A)$ onto $L_a^2(\mathbb{D})$, known as the Bergman projection, is well-defined. For more information on the theory of Bergman spaces, see [5].

For a function $f \in L^2(\mathbb{D}, \mathrm{d}A)$, we define the Toeplitz operator T_f from $L_a^2(\mathbb{D})$ to itself by $T_f(g) = P(fg)$ whenever fg is in $L^2(\mathbb{D}, \mathrm{d}A)$. The function f is called the symbol of the Toeplitz operator T_f. From this definition, it is clear that bounded analytic functions are within the domain of T_f, making T_f densely defined on $L_a^2(\mathbb{D})$. Furthermore, if the symbol f is bounded on $\mathbb{D}$, then T_f is bounded and $||T_f|| \le ||f||_\infty$.

Over the past 50 years, various algebraic properties of Toeplitz operators have been extensively studied. Nevertheless, the problem of describing the commutant of a given Toeplitz operator, i.e., the set of all Toeplitz operators that commute with it in the sense of composition, remains largely unresolved. Very little is known about when $T_fT_g = T_gT_f$ for "general symbols" f and g. For more information on the commutativity of Toeplitz operators on $L_a^2(D)$ and the results obtained so far for certain specific classes of symbols, see [1–4,6,7,9,10,12,14–16].

Dedicated to Rao V. Nagisetty (1938–2024).

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In this work, we address the class of so-called quasihomogeneous Toeplitz operators. A symbol f is said to be quasihomogeneous of degree p (an integer) if $f(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function. In this case, the associated Toeplitz operator T_f is also referred to as a quasihomogeneous Toeplitz operator of degree p (see [4,9,11]). The motivation for studying this family of symbols is that $L^2(\mathbb{D}, dA)$ can be expressed as $L^2(\mathbb{D}, dA) = \oplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}$, where $\mathcal{R}$ is the space of square-integrable radial functions on $[0, 1)$ with respect to the measure $r dr$. Thus, every function $f \in L^2(\mathbb{D}, dA)$ has the polar decomposition $f(z) = f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r)$, where f_k's are radial functions. We believe that studying quasihomogeneous Toeplitz operators will help us characterize the commutant of Toeplitz operators with more general symbols.

This present work is motivated by [2, Theorem 3.1, p. 52]. For our purposes, we will state and summarize this theorem as follows:

Theorem 1. *Let $\phi(r) = r^{(2M+1)p}$ and $\psi(r) = r^{(2N+1)s}$, where $p < s$ and M and N are integers greater or equal to 1. Suppose there exist $m, l \in \mathbb{N}$, and nontrivial radial functions f and g such that the following two hypotheses are satisfied*:

$$\begin{cases} (H1) \quad T_{e^{im\theta}f} + T_{e^{il\theta}g} \text{ commutes with } T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}, \\ (H2) \quad 1 \le p < s, \quad 1 \le m < l, \quad \text{and} \quad l + p = m + s. \end{cases}$$

Then, $m = p$, $l = s$, and

$$T_{e^{im\theta}f} + T_{e^{il\theta}g} = c(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi})$$

for some constant c.

Our goal here is to replace the radial components of the symbols $e^{ip\theta}\phi$ and $e^{is\theta}\psi$ in Theorem 1, specifically, $\phi(r) = r^{(2M+1)p}$ and $\psi(r) = r^{(2N+1)s}$, with $\phi(r) = r^n$ and $\psi(r) = r^d$, where n and d are in $\mathbb{N}$. In other words, we aim to eliminate the condition that the power of r in ϕ and ψ is an odd number multiple of the quasihomogeneous degrees p and s, respectively. Despite this modification, we will still be able to apply the result in [8] regarding the existence of roots for Toeplitz operators. This result states that there exist radial functions $\widetilde{\phi}$ and $\widetilde{\psi}$ such that $T_{e^{ip\theta}r^n} = (T_{e^{i\theta}\widetilde{\phi}})^p$ and $T_{e^{is\theta}r^d} = (T_{e^{i\theta}\widetilde{\psi}})^s$. Now, if hypotheses $(H1)$ and $(H2)$ are satisfied, then [9, Remark 2] implies that for every vector z^k of the orthogonal basis of $L^2_a(\mathbb{D})$, we have

$$T_{e^{im\theta}f} T_{e^{ip\theta}r^n}(z^k) = T_{e^{ip\theta}r^n} T_{e^{im\theta}f}(z^k), \tag{1}$$

$$T_{e^{il\theta}g} T_{e^{is\theta}r^d}(z^k) = T_{e^{is\theta}r^d} T_{e^{il\theta}g}(z^k), \tag{2}$$

$$(T_{e^{im\theta}f} T_{e^{is\theta}r^d} + T_{e^{il\theta}g} T_{e^{ip\theta}r^n})(z^k) = (T_{e^{is\theta}r^d} T_{e^{im\theta}f} + T_{e^{ip\theta}r^n} T_{e^{il\theta}g})(z^k). \tag{3}$$

From equations (1) and (2), we can conclude that $T_{e^{im\theta}f}$ and $T_{e^{il\theta}g}$ commute with $T_{e^{ip\theta}r^n}$ and $T_{e^{is\theta}r^d}$, respectively. Therefore, [9, Proposition 2 and Lemma 2] imply that

$$T_{e^{im\theta}f} = c_1(T_{e^{i\theta}\widetilde{\phi}})^m \tag{4}$$

and

$$T_{e^{il\theta}g} = c_2(T_{e^{i\theta}\widetilde{\psi}})^l \tag{5}$$

for some constants c_1 and c_2. To avoid the trivial case where the operators are zero, we assume from now on that c_1 and c_2 are nonzero constants. Finally, considering the commutator $[T, S] = TS - ST$ of two operators T and S, equation (3) can be written as

$$c_1[(T_{e^{i\theta}\widetilde{\phi}})^m, T_{e^{is\theta}r^d}](z^k) = c_2[T_{e^{ip\theta}r^n}, (T_{e^{i\theta}\widetilde{\psi}})^l](z^k), \quad \text{for all } k \ge 0. \tag{6}$$

Important comments

(i) If $T_{e^{ip\theta}r^n}$ commutes with $T_{e^{is\theta}r^d}$, then [9, Proposition 2 and Lemma 2] imply that $T_{e^{i\theta}\widetilde{\psi}} = cT_{e^{i\theta}\widetilde{\phi}}$ for some constant c. Moreover, [9, Corollary 1] indicates that all four Toeplitz operators $T_{e^{ip\theta}r^n}$, $T_{e^{is\theta}r^d}$, $T_{e^{im\theta}f}$, and $T_{e^{il\theta}g}$

commute with each other. Consequently, they are all of the form a constant multiple of a single Toeplitz operator, which is $T_{e^{i\theta}\widetilde{\phi}}$. Thus, without loss of generality, we assume that $[T_{e^{ip\theta}r^n}, T_{e^{is\theta}r^d}] \neq 0$.

(ii) The case $p = s$ (and similarly, $l = m$) has been extensively studied and fully solved. For detailed discussions, see [4,7,12].

(iii) We will demonstrate that if $(H1)$ and $(H2)$ hold, then $m = p$, $l = s$, which implies that the constants c_1 and c_2 appearing in equation (6) are equal. Specifically, if $m = p$ (or equivalently, $l = s$), then $(H2)$ ensures that $l = s$ (or equivalently, $m = p$). Moreover, equations (4) and (5) imply that $T_{e^{im\theta}f} = c_1 T_{e^{ip\theta}r^n}$ and $T_{e^{il\theta}g} = c_2 T_{e^{is\theta}r^d}$, where c_1 and c_2 are constants. Consequently, equation (6) simplifies to

$$c_1[T_{e^{ip\theta}r^n}, T_{e^{is\theta}r^d}](z^k) = c_2[T_{e^{ip\theta}r^n}, T_{e^{is\theta}r^d}](z^k), \quad \text{for all } k \geq 0.$$

Thus, $c_1 = c_2$ assuming that $[T_{e^{ip\theta}r^n}, T_{e^{is\theta}r^d}] \neq 0$.

2 Preliminaries and tools

Radial functions in $L^1(\mathbb{D}, dA)$ can be viewed as functions in $L^1([0,1), rdr)$. For a function $\phi \in L^1([0,1), rdr)$, we define its Mellin transform, denoted $\widehat{\phi}$, by

$$\widehat{\phi}(z) = \int_0^1 \phi(r) r^{z-1} r dr.$$

It is well known that for functions $\phi \in L^1([0,1) rdr)$, the Mellin transform $\widehat{\phi}$ is analytic on the right-half plane $\{z \in \mathbb{C} : \Re z > 2\}$ and is continuous and bounded on $\{z \in \mathbb{C} : \Re z \geq 2\}$.

The Mellin transform, which is related to the Laplace transform via the change of variable $r = e^{-t}$, is a valuable tool in studying quasihomogeneous Toeplitz operators. Indeed, quasihomogeneous Toeplitz operators act on the vectors of the orthogonal basis of $L_a^2(\mathbb{D})$ as shift operators with analytic weight, and this weight involves the Mellin transform of the symbol. We have the following lemma from [10, Lemma 1, p. 883].

Lemma 1. *Let $\phi \in L^1([0,1), rdr)$ and let p be a non-negative integer. Then, for every integer $k \geq 0$, we have*

$$T_{e^{ip\theta}\phi}(z^k) = 2(k+p+1)\widehat{\phi}(2k+p+2) z^{k+p}.$$

Another important property of the Mellin transform is that $\widehat{\phi}$ is uniquely determined by its values on any set of integers satisfying the Müntz-Szász (or Blaschke) condition. In our calculations, we often determine ϕ by knowing its Mellin transform $\widehat{\phi}$ on arithmetic sequences. We have the following classical theorem [13, p. 102].

Theorem 2. *Suppose that f is a bounded analytic function on the right-half plane $\{z \in \mathbb{C} : \Re z > 0\}$ that vanishes at a set of pairwise distinct points $d_1, d_2, \ldots,$ where*

(i) $\inf\{|d_n|\} > 0$ *and*

(ii) $\sum_{n\geq 1} \Re\left(\frac{1}{d_n}\right) = \infty$.

Then, f vanishes identically on $\{z \in \mathbb{C} : \Re z > 0\}$.

The following lemma plays a key role in our proof of the main result. Specifically, at a certain stage in the proof, we need to determine when the quotient of four Gamma functions is a rational function. We omit the proof of the lemma, which is a slight modification of [4, Theorem 3, p. 197–198].

Lemma 2. *Let a, b, c, d be non-negative integers such that $a + b - c - d = \lambda$, and let $\delta \in \mathbb{N}$. Define the function H to be*

$$H(z) = \frac{\Gamma\left(\frac{z}{2\delta} + \frac{a}{2\delta}\right)\Gamma\left(\frac{z}{2\delta} + \frac{b}{2\delta}\right)}{\Gamma\left(\frac{z}{2\delta} + \frac{c}{2\delta}\right)\Gamma\left(\frac{z}{2\delta} + \frac{d}{2\delta}\right)}.$$

Then, H is a rational function if and only if 2δ divides λ and one of the numbers $a - c$ or $a - d$.

In [4, Theorem 3, p. 197–198], the authors assume $a + b - c - d = -1$ rather than $a + b - c - d = \lambda$ as in our version above. However, the proof remains exactly the same as stated in [4, p. 205].

3 Main results

To ensure clarity and effectively persuade readers who may not be entirely familiar with the calculations in our proofs, we will begin by meticulously describing the case when $s = 2p$. This approach aims to enhance understanding and reinforce confidence in the validity of our results. Moreover, the proof of the general case (Theorem 4) is fundamentally based on this key scenario.

Theorem 3. *Let $\phi(r) = r^n$ and $\psi(r) = r^d$ with $p < s, n, d \in \mathbb{N}$. Suppose there exist $m, l \in \mathbb{N}$ and nontrivial radial functions f, g such that hypotheses (H1) and (H2) of Theorem 1 are satisfied. If $s = 2p$, then $m = p$, $l = s$, and*

$$T_{e^{im\theta}f} + T_{e^{is\theta}g} = c(T_{e^{ip\theta}r^n} + T_{e^{is\theta}r^d}),$$

for some constant c.

For simplicity, we will adopt the following notation in the proof: we will write $\equiv$ instead of $=$ when the quantity on the left-hand side of the equation is equal to a constant multiple of the quantity on the right-hand side.

Proof. Let $p, s, m, l, d, n \in \mathbb{N}^*$. Suppose that hypotheses $(H1)$ and $(H2)$ are satisfied. In this case, equation (6) implies

$$\begin{aligned}
&\frac{2k + 2s + 2}{2k + s + d + 2} \prod_{j=0}^{m-1} (2k + 2s + 2j + 4)\widehat{\widehat{\phi}}(2k + 2s + 2j + 3) \\
&\quad - \frac{2k + 2m + 2s + 2}{2k + 2m + s + d + 2} \prod_{j=0}^{m-1} (2k + 2j + 4)\widehat{\widehat{\phi}}(2k + 2j + 3) \\
&\equiv \frac{2k + 2l + 2p + 2}{2k + 2l + p + n + 2} \prod_{j=0}^{l-1} (2k + 2j + 4)\widehat{\widehat{\psi}}(2k + 2j + 3) \\
&\quad - \frac{2k + 2p + 2}{2k + p + n + 2} \prod_{j=0}^{l-1} (2k + 2p + 2j + 4)\widehat{\widehat{\psi}}(2k + 2p + 2j + 3).
\end{aligned}$$

Set $z = 2k + 2$. By applying [7, Theorem 14, p. 1473], it follows that

$$\frac{(z+2m+2p)(z+p+n)(z+2p+p+n)\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{(z+s+d)(z+2p)(z+2m+p+n)(z+2p+2m+p+n)\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$$

$$-\frac{1}{z+2m+s+d}\cdot\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$$

$$\equiv\frac{(z+2l)\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{(z+2m)\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}\cdot\frac{1}{z+2l+p+n}$$

$$-\frac{z\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{(z+p+n)(z+2m+s+d)\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}.$$

This above equation is equivalent to

$$R_1(z)\cdot\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}\equiv R_2(z)\cdot\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}-R_3(z)\cdot\frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)},\tag{7}$$

where R_1, R_2, and R_3 are rational functions in z. At this stage, we need to consider the case "$l < s$ and $m < p$" separately from the case "$l > s$ and $m > p$."

Case I. $l < s$ and $m < p$.

Our aim here is to show that the function $\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$ is rational. We will proceed by contradiction. Assume, for the sake of contradiction, that this function is not rational. We consider the following two subcases:

(1) Suppose the functions

$$\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)\quad\text{and}\quad\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)$$

have the same poles. According to Lemma 2, this implies that $2s$ divides $2m-s-d$. Hence, there exists $N\in\mathbb{N}$ such that $s+d=2m+2sN$. Now, let A be a sufficiently large integer. For such an A, the number $-2p(2A+1)$ is not a pole for the functions R_1, R_2, or R_3. However, it is a zero of both the left-hand side of equation (7) and the function $\frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}$. Consequently, it is also a zero of the function $\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}$. This implies that there exists $B\in\mathbb{N}$ such that $-2p(2A+1)=-2sB-2l-s-d$, which leads to $2s(A-B)=2m+s+d$. We deduce that $2s$ divides $2m+s+d$, and consequently, $2s$ divides $4m$. This implies that p divides m, which is a contradiction because $m<p$.

(2) Suppose that the function $\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}$ is not rational. Dividing both sides of equation (7) by $\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)$ yields

$$R_1(z)\cdot\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}$$

$$\equiv R_2(z)\cdot\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}-\frac{R_3(z)}{\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}.$$

So, the following functions must be rational:

$$\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}\quad\text{and}\quad\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}.$$

Applying Lemma 2, we find that $2s$ must divide $s+d$, and hence, the right-hand side of equation (7) is rational. Therefore, it follows that the function $\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$ is rational as well. This leads to a contradiction.

Since we proved that the function $\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$ is rational, Lemma 2 implies that $2p$ must divide $p+n$. Consequently, n must be an odd number multiple of p. Similarly, from equation (7), the functions

$$\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}\quad\text{and}\quad\frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}$$

must also be rational. Applying Lemma 2 again, we find that $2s$ must divide $s+d$. Therefore, d must be an odd number multiple of s. Thus, Theorem 1 completes the proof.

Case II. $l > s$ and $m > p$.

Observe that if p divides m, then the function $\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)}$ is rational. In this case, we have the following two possibilities:

(1) If $m=2Np$, then the function $\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}$ is rational. According to Lemma 2, $2s$ must divide $2m-s-d$.

Since $2s$ divides $2m$, it follows that $2s$ must also divide $s+d$. Thus, there exists an integer $M\in\mathbb{N}$ such that $2sM=s+d$. This implies that d is an odd number multiple of s. Next, equation (7) becomes

$$R_1(z)\prod_{i=0}^{2N-1}\frac{z+2pi}{z+p+n+2pi}\equiv R_2(z)\prod_{i=0}^{M-1}\frac{z+2si}{z+2l+2si}-R_3(z)\prod_{i=0}^{M-1}\frac{z+2p+2si}{z+2m+2si}.$$

The point $-p-n-2p(2N-1)$ is a pole of the right-hand side of the equation above. This pole can only arise from $-2l-p-n$ or $-p-n$, as all the other poles are multiples of $2p$. For the equation to hold, this pole should be canceled. Thus, $2p$ divides $p+n$, which implies that n is an odd number multiple of p. Hence, Theorem 1 completes the proof.

(2) Assume $m = (2N+1)p$. Then, both

$$\frac{\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)} \quad \text{and} \quad \frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}$$

are rational. This leads to

$$R_1(z)\prod_{i=0}^{2N}\frac{z+2pi}{z+p+n+2pi} \equiv R_2(z)\prod_{i=0}^{N}\frac{z+2si}{z+s+d+2si} - R_3(z)\prod_{i=0}^{N-1}\frac{z+2p+2si}{z+2p+s+d+2si}. \tag{8}$$

We shall argue by contradiction. Suppose $2s$ does not divide $s+d$. Then, the two products on the right-hand side of the previous equation cannot be simplified or canceled. Consider the point $-4pN+4p$. This point is a zero of the function $R_3(z)\prod_{i=0}^{N-1}\frac{(z+2p+2si)}{(z+2p+s+d+2si)}$. Hence, there must exist $\exists\ i \in \{0,\dots,N-1\}$ such that $-4pN+4p = -2p-2si$, which leads to a contradiction because we are assuming that $s = 2p$. Thus, our assumption that $2s$ does not divide $s+d$ must be incorrect. Therefore, $2s$ does divide $s+d$, which implies that d is an odd number multiple of s. Now, let us assume that $2p$ does not divide $p+n$. The point $-p-n-2p(2N)$ would then be a pole on the left-hand side of equation (8), but all poles on the right-hand side are multiples of $2p$, which is a contradiction. Therefore, $2p$ must divide $p+n$, which implies that n is an odd number multiple of p. Hence, Theorem 1 completes the proof.

The proof of the case where $\frac{m}{p} \notin \mathbb{N}$ is analogous to **Case I**. □

For the following result, we will relax the condition previously required in Theorem 3, specifically $s = 2p$.

Theorem 4. *Let $\phi(r) = r^n$ and $\psi(r) = r^d$ with $p < s$ and $n, d \in \mathbb{N}$. Suppose there exist $m, l \in \mathbb{N}$ and nontrivial radial functions f, g such that hypotheses (H1) and (H2) are satisfied. Then, $m = p$, $l = s$, and*

$$T_{e^{im\theta}f} + T_{e^{is\theta}g} = c(T_{e^{ip\theta}r^n} + T_{e^{is\theta}r^d})$$

for some constant c.

Proof. By setting $z = 2k+2$ in equation (6) and applying [7, Theorem 14, p. 1473], we obtain

$$\begin{aligned}
&\frac{z+2s}{z+s+d}\cdot\frac{(z+2s+2m)\Gamma\left(\frac{z+2s+2m}{2p}\right)\Gamma\left(\frac{z+2s+p+n}{2p}\right)}{(z+2s)\Gamma\left(\frac{z+2s}{2p}\right)\Gamma\left(\frac{z+2s+2m+p+n}{2p}\right)} \\
&\quad - \frac{z+2m+2s}{z+2m+s+d}\cdot\frac{(z+2m)\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{z\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)} \\
&\equiv \frac{z+2l+2p}{z+2l+p+n}\cdot\frac{(z+2l)\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{z\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)} \\
&\quad - \frac{z+2p}{z+p+n}\cdot\frac{(z+2p+2l)\Gamma\left(\frac{z+2p+2l}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{(z+2p)\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2p+2l+s+d}{2s}\right)}.
\end{aligned}$$

By (H2), there exist $\alpha, \beta \in \mathbb{N}$ such that $m + \beta = l$ and $p + \alpha = s$. Since $l + p = m + s$, we have $m + \beta + p = m + p + \alpha$. Therefore, $\beta = \alpha$. Let $\alpha \in \mathbb{N}$ be such that $p + \alpha = s, m + \alpha = l$. Then, equation (6) becomes

$$\begin{aligned}
&\frac{(z+2s)}{(z+s+d)} \cdot \frac{(z+2s+2m)(z+2\alpha+2m)(z+2\alpha+p+n)\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)}{(z+2s)(z+2\alpha)(z+2\alpha+2m+p+n)\Gamma\left(\frac{z+2\alpha}{2p}\right)\Gamma\left(\frac{z+2\alpha+2m+p+n}{2p}\right)} \\
&\quad - \frac{(z+2m+2s)}{(z+2m+s+d)} \cdot \frac{(z+2m)\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{z\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)} \\
&\equiv \frac{(z+2l+2p)}{(z+2l+p+n)} \cdot \frac{(z+2l)\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{z\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)} \\
&\quad - \frac{(z+2p)}{(z+p+n)} \cdot \frac{(z+2p+2l)(z+2m)\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{(z+2p)(z+2m+s+d)\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\frac{(z+2\alpha+p+n)(z+2m+s+d)(z+2l)}{(z+s+d)(z+2l+p+n)(z+2\alpha)}\left[\frac{\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)}{\Gamma\left(\frac{z+2\alpha}{2p}\right)\Gamma\left(\frac{z+2\alpha+2m+p+n}{2p}\right)}\right. \\
&\qquad \left. - \frac{(z+s+d)(z+2\alpha)\Gamma\left(\frac{z+2l}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{(z+2\alpha+p+n)(z)\Gamma\left(\frac{z}{2s}\right)\Gamma\left(\frac{z+2l+s+d}{2s}\right)}\right] \\
&\equiv \frac{z+2m}{z}\left[\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)} - \frac{z\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{(z+p+n)\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}\right].
\end{aligned}$$

This simplifies to

$$H(z)F(z+2\alpha) \equiv F(z), \tag{9}$$

where

$$F(z) = \frac{z+2m}{z}\left[\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z}{2p}\right)\Gamma\left(\frac{z+2m+p+n}{2p}\right)} - \frac{z\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{(z+p+n)\Gamma\left(\frac{z+2p}{2s}\right)\Gamma\left(\frac{z+2m+s+d}{2s}\right)}\right]$$

and

$$H(z) = \frac{(z+2\alpha+p+n)(z+2m+s+d)}{(z+2l+p+n)(z+s+d)}.$$

It is easy to see that the function F has infinitely many poles. Specifically, F has poles at the points $-2sA - 2m, -2sB - 2p - s - d, -2pC - 2m, -2pD - p - n$ for sufficiently large integers $A, B, C, D \in \mathbb{N}$. These integers are chosen large enough to ensure that these poles are not canceled out by the zeros of the function $H(z)$. These poles arise from the terms $\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)$ and $\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)$. Additionally, equation (9) indicates that these points are also poles of the function $F(z+2\alpha)$. We then consider the following two situations:

Situation 1: The poles of $\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)$ arise from $\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)$. Similarly, the poles of $\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)$ arise from $\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)$. This implies that the functions

$$\frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)} \quad \text{and} \quad \frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)}$$

are rational functions. By applying Lemma 2, we conclude that s must divide $p-\alpha$ and p must divide 2α. Hence, $p=\alpha$. Since $s=p+\alpha$, it follows that $s=2p$. Therefore, Theorem 3 completes the proof.

Situation 2: The poles of $\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)$ arise from $\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)$. Similarly, the poles of $\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)$ arise from $\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)$. This implies that the functions

$$\frac{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}{\Gamma\left(\frac{z+2\alpha+2m}{2p}\right)\Gamma\left(\frac{z+2\alpha+p+n}{2p}\right)} \quad \text{and} \quad \frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}$$

are rational functions. Furthermore, the function $\frac{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)}$ is also rational. Consequently, when multiplying equation (9) with the ratio $\frac{\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z+2m}{2p}\right)\Gamma\left(\frac{z+p+n}{2p}\right)}$, we obtain the function

$$\frac{\Gamma\left(\frac{z+2\alpha+2m}{2s}\right)\Gamma\left(\frac{z+s+d}{2s}\right)}{\Gamma\left(\frac{z+2m}{2s}\right)\Gamma\left(\frac{z+2p+s+d}{2s}\right)},$$

which is also rational. According to Lemma 2, this implies that s divides $\alpha-p$. If $\alpha>p$, then $s=p+\alpha\leq\alpha-p$, which is not possible. Conversely, if $\alpha<p$, then $s=p+\alpha\leq p-\alpha$, which is also not possible. Thus, $p=\alpha$, which implies $s=2p$. Hence, Theorem 3 completes the proof. □

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