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## ON $n$ -SEMIPRIMARY IDEALS AND $n$ -PSEUDO VALUATION DOMAINS

DAVID F. ANDERSON AND AYMAN BADAWI

ABSTRACT. Let  $R$  be a commutative ring with  $1 \neq 0$  and  $n$  a positive integer. A proper ideal  $I$  of  $R$  is an  $n$ -semiprimary ideal of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in R$ , then  $x^n \in I$  or  $y^n \in I$ . Let  $R$  be an integral domain with quotient field  $K$ . A proper ideal  $I$  of  $R$  is an  $n$ -powerful ideal of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in K$ , then  $x^n \in R$  or  $y^n \in R$ ; and  $I$  is an  $n$ -powerful semiprimary ideal of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in K$ , then  $x^n \in I$  or  $y^n \in I$ . If every prime ideal of  $R$  is an  $n$ -powerful semiprimary ideal of  $R$ , then  $R$  is an  $n$ -pseudo-valuation domain ( $n$ -PVD). In this paper, we study the above concepts and relate them to several generalizations of pseudo-valuation domains.

### 1. INTRODUCTION

Let  $R$  be a commutative ring with  $1 \neq 0$  and  $n$  a positive integer. Recall that an ideal  $I$  of  $R$  is a *semiprimary ideal* of  $R$  if  $\sqrt{I}$  is a prime ideal of  $R$ . In this paper, we introduce and study  $n$ -semiprimary ideals (resp.,  $n$ -powerful semiprimary ideals in integral domains), where a proper ideal  $I$  of  $R$  is  $n$ -semiprimary (resp.,  $n$ -powerful semiprimary) if whenever  $x^n y^n \in I$  for  $x, y \in R$  (resp.,  $x, y \in K$ , the quotient field of  $R$ ), then  $x^n \in I$  or  $y^n \in I$ . These concepts generalize prime ideals and are generalized by semiprimary ideals. We also investigate several other “ $n$ ” generalizations obtained by replacing  $x$  with  $x^n$  in the definition.

In Section 2, we give some basic properties of  $n$ -semiprimary ideals. For example, we show that an  $n$ -semiprimary ideal is semiprimary, and the converse holds when  $R$  is Noetherian. We also show that an  $n$ -semiprimary ideal is  $m$ -semiprimary for every integer  $m \geq n$ . In Section 3, we characterize  $n$ -semiprimary ideals in several classes of commutative rings. In particular, we investigate  $n$ -semiprimary ideals in zero-dimensional commutative rings, Dedekind domains, valuation domains, and idealizations. In Section 4, we study  $n$ -powerful semiprimary ideals in integral domains and introduce  $n$ -pseudo-valuation domains ( $n$ -PVDs), a generalization of pseudo-valuation domains (PVDs). We also study  $n$ -valuation domains ( $n$ -VDs). In the final section, Section 5, we introduce pseudo  $n$ -valuation domains ( $Pn$ VDs), another generalization of PVDs. Many examples are given throughout the paper to illustrate the theory.

Throughout,  $R$  will be a commutative ring with  $1 \neq 0$ ,  $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$  for  $I$  an ideal of  $R$ , ideal of nilpotent elements  $\text{nil}(R) = \sqrt{\{0\}}$ , group of units  $U(R)$ , (Krull) dimension  $\dim(R)$ , and characteristic  $\text{char}(R)$ . An overring

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of an integral domain  $R$  with quotient field  $K$  is a subring of  $K$  containing  $R$ , and we denote the integral closure of  $R$  (in  $K$ ) by  $\bar{R}$ . In particular, if  $I$  is an ideal of  $R$ , then  $(I : I) = \{x \in K \mid xI \subseteq I\}$  is an overring of  $R$ . Other definitions will be given throughout the paper as needed. As usual,  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Z}_n$ ,  $\mathbb{F}_{p^n}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  will denote the set of positive integers, the rings of integers and integers mod  $n$ , the finite field with  $p^n$  elements, and the fields of rational numbers, real numbers, and complex numbers, respectively. For any undefined terminology, see [23], [26], [27], or [28].

## 2. BASIC PROPERTIES OF $n$ -SEMIPRIMARY IDEALS

In this section, we give some basic properties of  $n$ -semiprimary ideals. We begin with the definition.

**Definition 2.1.** Let  $I$  be a proper ideal of a commutative ring  $R$  and  $n$  a positive integer. Then  $I$  is an  $n$ -semiprimary ideal of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in R$ , then  $x^n \in I$  or  $y^n \in I$ .

Note that a 1-semiprimary ideal is just a prime ideal. For convenience, call a commutative ring  $R$  an  $n$ -ring if  $x^n y^n = 0$  for  $x, y \in R$  implies  $x^n = 0$  or  $y^n = 0$ . Then a 1-ring is just an integral domain,  $R$  is an  $n$ -ring if and only if  $\{0\}$  is an  $n$ -semiprimary ideal of  $R$ , and  $R/I$  is an  $n$ -ring if and only if  $I$  is an  $n$ -semiprimary ideal of  $R$ . We start with some elementary results that follow directly from the definitions.

**Theorem 2.2.** Let  $I$  be a proper ideal of a commutative ring  $R$ .

- (a) Let  $I$  be an  $n$ -semiprimary ideal of  $R$ . Then  $I$  is an  $mn$ -semiprimary ideal of  $R$  for every positive integer  $m$ . (See Theorem 2.14 for a stronger result.)
- (b) Let  $J \subseteq I$  be proper ideals of  $R$ . Then  $I$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $I/J$  is an  $n$ -semiprimary ideal of  $R/J$ .
- (c) Let  $I$  be an  $n$ -semiprimary ideal of  $R$  and  $S$  a multiplicatively closed subset of  $R$  with  $I \cap S = \emptyset$ . Then  $I_S$  is an  $n$ -semiprimary ideal of  $R_S$ .

We next show that an  $n$ -semiprimary ideal is indeed semiprimary.

**Theorem 2.3.** Let  $I$  be an  $n$ -semiprimary ideal of a commutative ring  $R$ . Then  $\sqrt{I}$  is a prime ideal of  $R$  and  $x^n \in I$  for every  $x \in \sqrt{I}$ . In particular,  $I$  is a semiprimary ideal of  $R$ , and  $x \in \sqrt{I}$  if and only if  $x^n \in I$ .

*Proof.* Let  $xy \in \sqrt{I}$  for  $x, y \in R$ . Then there is a positive integer  $k$  such that  $(x^k)^n (y^k)^n = (xy)^{kn} \in I$ . Thus  $x^{kn} = (x^k)^n \in I$  or  $y^{kn} = (y^k)^n \in I$  since  $I$  is an  $n$ -semiprimary ideal of  $R$ . Hence  $x \in \sqrt{I}$  or  $y \in \sqrt{I}$ ; so  $\sqrt{I}$  is a prime ideal of  $R$ . Let  $x \in \sqrt{I}$  and  $m$  be the least positive integer such that  $x^{mn} \in I$ . Then  $x^n (x^{m-1})^n = x^n x^{(m-1)n} = x^{mn} \in I$ , and thus  $x^n \in I$  or  $x^{(m-1)n} \in I$  since  $I$  is an  $n$ -semiprimary ideal of  $R$ . Hence  $m = 1$ ; so  $x^n \in I$ . The ‘‘in particular’’ statement is clear.  $\square$

The following is an example of a semiprimary ideal of a commutative ring  $R$  that is not an  $n$ -semiprimary ideal for any positive integer  $n$ . Note that  $R$  is *not* Noetherian. In fact, Corollary 2.6 shows that semiprimary ideals in a commutative Noetherian ring are  $n$ -semiprimary for all large  $n$ .

**Example 2.4.** Let  $R = \mathbb{Z}_2[\{X_n\}_{n=1}^\infty]$  and  $I = (\{X_n^n\}_{n=1}^\infty)$ . Then  $\sqrt{I} = (\{X_n\}_{n=1}^\infty)$  is a prime ideal of  $R$ ; so  $I$  is a semiprimary ideal of  $R$ . However,  $I$  is not an  $n$ -semiprimary ideal of  $R$  for any positive integer  $n$  since  $X_{2n}^n X_{2n}^n = X_{2n}^{2n} \in I$ , but  $X_{2n}^n \notin I$ .

The next theorem gives a sufficient condition for a semiprimary ideal to be an  $n$ -semiprimary ideal. As a consequence,  $n$ -absorbing semiprimary ideals are  $n$ -semiprimary and semiprimary ideals in commutative Noetherian rings are  $n$ -semiprimary for all large  $n$ .

**Theorem 2.5.** *Let  $I$  be a proper ideal of a commutative ring  $R$  such that  $P = \sqrt{I}$  is a prime ideal of  $R$  and  $P^n \subseteq I$  for a positive integer  $n$ . Then  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$ . In particular,  $Q^n$  is an  $m$ -semiprimary ideal of  $R$  for every prime ideal  $Q$  of  $R$  and integer  $m \geq n$ .*

*Proof.* Let  $x^n y^n \in I \subseteq P$  for  $x, y \in R$ . Then  $x \in P$  or  $y \in P$ . Thus  $x^n \in P^n \subseteq I$  or  $y^n \in P^n \subseteq I$ , and hence  $I$  is an  $n$ -semiprimary ideal of  $R$ . Moreover,  $P^m \subseteq P^n \subseteq I$  for every integer  $m \geq n$ ; so  $I$  is also an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$ . The ‘‘in particular’’ statement is clear.  $\square$

**Corollary 2.6.** *Let  $I$  be a semiprimary ideal of a commutative Noetherian ring  $R$ . Then there is a positive integer  $n$  such that  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$ .*

*Proof.* Since  $I$  is a semiprimary ideal of  $R$ ,  $P = \sqrt{I}$  is a prime ideal of  $R$ , and  $P^n \subseteq I$  for some positive integer  $n$  since  $P$  is finitely generated. Thus  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$  by Theorem 2.5.  $\square$

Recall ([15], [9]) that a proper ideal  $I$  of a commutative ring  $R$  is an  $n$ -absorbing ideal of  $R$  if whenever  $x_1 \cdots x_{n+1} \in I$  for  $x_1, \dots, x_{n+1} \in R$ , then the product of  $n$  of the  $x_i$ 's is in  $I$  (for a related concept, also see [10]). Both  $n$ -semiprimary and  $n$ -absorbing ideals generalize prime ideals, but in rather different ways. An  $n$ -semiprimary ideal need not be an  $n$ -absorbing ideal (see Example 2.9); and an  $n$ -absorbing ideal need not be  $n$ -semiprimary since, for example,  $(6)$  is a 2-absorbing ideal of  $\mathbb{Z}$ , but not a 2-semiprimary ideal since  $\sqrt{(6)} = (6)$  is not a prime ideal of  $\mathbb{Z}$ . However, we next show that if  $\sqrt{I}$  is a prime ideal, then an  $n$ -absorbing ideal  $I$  is  $n$ -semiprimary.

**Corollary 2.7.** *Let  $I$  be an  $n$ -absorbing ideal of a commutative ring  $R$ . If  $\sqrt{I}$  is a prime ideal of  $R$ , then  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$ . In particular, an  $n$ -absorbing ideal is  $n$ -semiprimary if and only if it is semiprimary.*

*Proof.* Let  $P = \sqrt{I}$  be a prime ideal of  $R$ . Then  $P^n = (\sqrt{I})^n \subseteq I$  since  $I$  is an  $n$ -absorbing ideal of  $R$  ([18], [22]). Thus  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$  by Theorem 2.5. The ‘‘in particular’’ statement now follows from Theorem 2.3.  $\square$

**Corollary 2.8.** *Let  $P_1 \subseteq \cdots \subseteq P_k$  be prime ideals of a commutative ring  $R$  and  $n_1, \dots, n_k$  positive integers. Then  $I = P_1^{n_1} \cdots P_k^{n_k}$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n_1 + \cdots + n_k$ .*

*Proof.* Note that  $\sqrt{I} = P_1$  is a prime ideal of  $R$  and  $P_1^n \subseteq P_1^{n_1} \cdots P_k^{n_k} = I$ , where  $n = n_1 + \cdots + n_k$ . Thus  $I$  is an  $m$ -semiprimary ideal of  $R$  for every integer  $m \geq n$  by Theorem 2.5.  $\square$

The converse of Theorem 2.5 need not be true, i.e., if  $I$  is an  $n$ -semiprimary ideal of  $R$  for some integer  $n \geq 2$ , then  $(\sqrt{I})^n$  need not be a subset of  $I$ . Let  $p \geq 2$  be a prime integer. In the following example, we show that there is a proper ideal  $I$  of a commutative ring  $R$  such that  $I$  is a  $p$ -semiprimary ideal of  $R$ , but  $(\sqrt{I})^p \not\subseteq I$ , and thus  $I$  is not a  $p$ -absorbing ideal of  $R$  ([18], [22]).

**Example 2.9.** Let  $p \geq 2$  be a prime integer,  $R = \mathbb{Z}_p[X, Y]$ , and  $I = (X^p, Y^p)$ . Then  $I$  is a proper ideal of  $R$  with prime ideal  $P = \sqrt{I} = (X, Y)$  and  $P^p \not\subseteq I$  since  $YX^{p-1} \notin I$ . Thus  $I$  is not a  $p$ -absorbing ideal of  $R$  ([18], [22]). Let  $f^p g^p \in I \subseteq (X, Y)$  for  $f, g \in R$ . Then  $f \in (X, Y)$  or  $g \in (X, Y)$ ; so  $f^p \in I$  or  $g^p \in I$ , and hence  $I$  is a  $p$ -semiprimary ideal of  $R$ .

Recall [19] that a proper ideal  $I$  of a commutative ring  $R$  is a *uniformly primary ideal* of  $R$  if there is a positive integer  $n$  such that whenever  $xy \in I$  for  $x, y \in R$ , then  $x \in I$  or  $y^n \in I$ . If  $I$  is a uniformly primary ideal of  $R$  for a positive integer  $n$ , then we say that  $I$  is an  *$n$ -primary ideal* of  $R$ . By the following theorem, an  $n$ -primary ideal is also  $n$ -semiprimary.

**Theorem 2.10.** *Let  $I$  be an  $n$ -primary ideal of a commutative ring  $R$ . Then  $I$  is an  $n$ -semiprimary ideal of  $R$ .*

*Proof.* Let  $x^n y^n \in I$  for  $x, y \in R$  with  $x^n \notin I$ , and let  $m$  be the least positive integer such that  $x^n y^m \in I$ . Then  $(x^n y^{m-1})y = x^n y^m \in I$ . Since  $x^n y^{m-1} \notin I$  and  $I$  is an  $n$ -primary ideal of  $R$ , we have  $y^n \in I$ . Thus  $I$  is an  $n$ -semiprimary ideal of  $R$ .  $\square$

In the following example, we show that there is a commutative ring  $R$  with ideals  $\{I_n\}_{n=2}^\infty$  such that every  $I_n$  is an  $n$ -semiprimary ideal of  $R$  with  $(\sqrt{I_n})^n \subseteq I_n$ , but  $I_n$  is not a primary ideal of  $R$ . In particular,  $I_n$  is not an  $m$ -primary ideal of  $R$  for any positive integer  $m$ .

**Example 2.11.** Let  $R = \mathbb{Z}_2[X, Y]$ . For every integer  $n \geq 2$ ,  $I_n = (XY, Y^n)$  is an ideal of  $R$  with prime ideal  $P = \sqrt{I_n} = (Y)$ . Thus  $I_n$  is an  $n$ -semiprimary ideal of  $R$  by Theorem 2.5 since  $P^n \subseteq I_n$ . However,  $YX \in I_n$ ,  $Y \notin I_n$ , and  $X^m \notin I_n$  for every positive integer  $m$ ; so  $I_n$  is not a primary ideal of  $R$ , and hence  $I_n$  is not an  $m$ -primary ideal of  $R$  for any positive integer  $m$ .

The next definition generalizes the “ $n$ -semiprimary” concept from elements to ideals.

**Definition 2.12.** Let  $I$  be a proper ideal of a commutative ring  $R$  and  $n$  a positive integer. Then  $I$  is a *strongly  $n$ -semiprimary ideal* of  $R$  if whenever  $J^n K^n \subseteq I$  for proper ideals  $J$  and  $K$  of  $R$ , then  $J^n \subseteq I$  or  $K^n \subseteq I$ .

A strongly 1-semiprimary ideal is just a prime ideal, a strongly  $n$ -semiprimary ideal is an  $n$ -semiprimary ideal, and a strongly  $n$ -semiprimary ideal is also strongly  $mn$ -semiprimary for every positive integer  $m$ . However, the following example shows that an  $n$ -semiprimary ideal need not be strongly  $n$ -semiprimary.

**Example 2.13.** Let  $R = \mathbb{Z}_2[X, Y]$  and  $I = (X^2, Y^2)$ . By Example 2.9,  $I$  is a 2-semiprimary ideal of  $R$  with prime ideal  $P = \sqrt{I} = (X, Y)$ . Clearly,  $P^2 P^2 = P^4 \subseteq I$ , but  $P^2 \not\subseteq I$ . Thus  $I$  is not a strongly 2-semiprimary ideal of  $R$ . Note that  $I$  is an  $n$ -semiprimary ideal of  $R$  for every integer  $n \geq 3$  by Theorem 2.5 since  $P^3 \subseteq I$ , and hence  $I$  is an  $n$ -semiprimary ideal of  $R$  for every integer  $n \geq 2$ .

We have already observed in Theorem 2.2 that an  $n$ -semiprimary ideal is also  $mn$ -semiprimary for every positive integer  $m$ . We next give a much stronger result.

**Theorem 2.14.** *Let  $I$  be an  $n$ -semiprimary ideal of a commutative ring  $R$ .*

- (a) *If  $x^m y^k \in I$  for  $x, y \in R$  and positive integers  $m$  and  $k$ , then  $x^n \in I$  or  $y^n \in I$ . In particular, if  $x^m \in I$  for  $x \in R$  and  $m$  a positive integer, then  $x^n \in I$ .*
- (b)  *$I$  is an  $m$ -semiprimary ideal of  $R$  for every positive integer  $m \geq n$ .*

*Proof.* (a) Let  $x^m y^k \in I$  for  $x, y \in R$ ; we may assume that  $m \geq k$ . Then  $(xy)^m = x^m y^m = (x^m y^k) y^{m-k} \in I$ . Thus  $xy \in \sqrt{I}$ ; so  $x^n y^n = (xy)^n \in I$  by Theorem 2.3. Hence  $x^n \in I$  or  $y^n \in I$  since  $I$  is an  $n$ -semiprimary ideal of  $R$ . The ‘‘in particular’’ statement is clear.

(b) Let  $x^m y^m \in I$  for  $x, y \in R$  with  $m \geq n$ . Then  $x^n \in I$  or  $y^n \in I$  by part (a). Thus  $x^m = x^{m-n} x^n \in I$  or  $y^m = y^{m-n} y^n \in I$  since  $m \geq n$ ; so  $I$  is an  $m$ -semiprimary ideal of  $R$ .  $\square$

An ideal may be  $n$ -semiprimary for many different values of  $n$ . We now make that statement more precise. For a proper ideal  $I$  of a commutative ring  $R$ , let  $W_R(I) = \{n \in \mathbb{N} \mid I \text{ is an } n\text{-semiprimary ideal of } R\}$  and  $\delta_R(I) = \min W_R(I)$  (let  $\delta_R(I) = \infty$  if  $W_R(I) = \emptyset$ ). Then  $W_R(I) = [\delta_R(I), \infty) \cap \mathbb{N}$  by Theorem 2.14(b).

### 3. $n$ -SEMIPRIMARY IDEALS IN SOME CLASSES OF RINGS

In this section, we study  $n$ -semiprimary ideals in several important classes of commutative rings. We have already observed in Corollary 2.6 that for commutative Noetherian rings, a semiprimary ideal is  $n$ -semiprimary for all large  $n$ . The first two results concern the case when  $\dim(R) = 0$ .

**Theorem 3.1.** *Let  $I \supseteq \text{nil}(R)$  be an ideal of a commutative ring  $R$  with  $\dim(R) = 0$ . Then  $I$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $I$  is a prime ideal of  $R$  (i.e.,  $I$  is a 1-semiprimary ideal of  $R$ ).*

*Proof.* A prime ideal is certainly  $n$ -semiprimary for every positive integer  $n$ . Conversely, we show that an  $n$ -semiprimary ideal  $I$  of  $R$  is a prime ideal of  $R$ . Let  $xy \in I$  for  $x, y \in R$ ; so  $x^n y^n \in I$ . Then  $x^n \in I$  or  $y^n \in I$ ; say  $x^n \in I$ . Since  $\dim(R) = 0$ , we have  $x = eu + w$  for an idempotent  $e \in R$ ,  $u \in U(R)$ , and  $w \in \text{nil}(R)$  [13, Corollary 1]. Thus  $x^n = (eu + w)^n = eu^n + a_1 eu^{n-1} w + a_2 eu^{n-2} w^2 + \cdots + a_{n-1} eu w^{n-1} + w^n = e(u^n + a_1 u^{n-1} w + a_2 u^{n-2} w^2 + \cdots + a_{n-1} u w^{n-1}) + w^n \in I$ , where the  $a_i$ 's are positive integers, and  $v = u^n + a_1 u^{n-1} w + a_2 u^{n-2} w^2 + \cdots + a_{n-1} u w^{n-1} \in U(R)$ . Hence  $x^n = (eu + w)^n = ev + w^n$  with  $w^n \in \text{nil}(R) \subseteq I$ . Thus  $ev = x^n - w^n \in I$ , and hence  $eu = (ev)(v^{-1}u) \in I$ . Thus  $x = eu + w \in I$ ; so  $I$  is a prime ideal of  $R$ .  $\square$

**Corollary 3.2.** *Let  $R$  be a commutative von-Neumann regular ring. Then a proper ideal  $I$  of  $R$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $I$  is a prime ideal of  $R$ .*

*Proof.* A commutative ring  $R$  is von Neumann regular if and only if  $\text{nil}(R) = \{0\}$  and  $\dim(R) = 0$  [26, page 5].  $\square$

However, if  $I$  is an  $n$ -semiprimary ideal of a zero-dimensional commutative ring  $R$  for some integer  $n \geq 2$  and  $\text{nil}(R) \not\subseteq I$ , then  $I$  need not be a prime ideal of  $R$ . We have the following example.

**Example 3.3.** Let  $R = \mathbb{Z}_4 \times \mathbb{Z}_2$ . Then  $\dim(R) = 0$  and  $I = \{0\} \times \mathbb{Z}_2$  is a 2-semiprimary ideal of  $R$  with  $\text{nil}(R) = \{0, 2\} \times \{0\} \not\subseteq I$ . However,  $I$  is not a prime ideal of  $R$ .

It is easy to determine the  $n$ -semiprimary ideals in a Dedekind domain  $R$  since every nonzero proper ideal of  $R$  is (uniquely) a product of prime (maximal) ideals [28, Theorem 6.16].

**Theorem 3.4.** *Let  $I$  be a nonzero proper ideal of a Dedekind domain  $R$ . Then  $I$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $I = P^k$ , where  $P = \sqrt{I}$  is a prime (maximal) ideal of  $R$  and  $n \geq k$ . Moreover,  $\delta_R(I) = n$  if and only if  $I = P^n$ .*

*Proof.* Let  $I$  be a nonzero proper ideal of a Dedekind domain  $R$ . Then  $\sqrt{I} = P$  is a prime (maximal) ideal if and only if  $I = P^k$  for some positive integer  $k$ . Thus by Theorem 2.3 and Theorem 2.5,  $I$  is  $n$ -semiprimary if and only if  $I = P^k$  for some positive integer  $k$ , where  $n \geq k$ . The ‘‘in particular’’ statement is clear.  $\square$

Next, we give a characterization of Dedekind domains in terms of 2-semiprimary ideals.

**Theorem 3.5.** *Let  $R$  be a Noetherian integral domain. Then the following statements are equivalent.*

- (1)  $R$  is a Dedekind domain.
- (2) If  $I$  is an ideal of  $R$  with  $\delta_R(I) = 2$ , then  $I = M^2$  for some maximal ideal  $M$  of  $R$ .

*Proof.* (1)  $\Rightarrow$  (2) This follows directly from Theorem 3.4.

(2)  $\Rightarrow$  (1) Let  $I$  be an ideal of  $R$  with  $M^2 \subseteq I \subsetneq M$  for a maximal ideal  $M$  of  $R$ . Then  $I$  is 2-semiprimary by Theorem 2.5 and not prime (maximal); so  $\delta_R(I) = 2$ . Thus  $I = M^2$  by hypothesis. Hence there are no ideals of  $R$  strictly between  $M$  and  $M^2$  for every maximal ideal  $M$  of  $R$ ; so  $R$  is a Dedekind domain by [28, Theorem 6.20].  $\square$

It is also easy to describe the  $n$ -semiprimary ideals in a valuation domain. Recall that every proper ideal in a valuation domain is semiprimary [23, Theorem 17.1(2)].

**Theorem 3.6.** *Let  $I$  be a proper ideal of a valuation domain  $R$  with  $P = \sqrt{I}$ .*

- (a)  $I$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $P^n \subseteq I$ .
- (b) If  $P$  is idempotent, then  $I$  is an  $n$ -semiprimary ideal of  $R$  if and only if  $I = P$ .
- (c) If  $P$  is not idempotent, then  $I$  is an  $n$ -semiprimary ideal of  $R$  for some positive integer  $n$ . Moreover, every ideal of  $R$  between  $P$  and the prime ideal directly below  $P$  is an  $n$ -semiprimary ideal for some positive integer  $n$ .

*Proof.* (a) If  $P^n \subseteq I$ , then  $I$  is  $n$ -semiprimary by Theorem 2.5. Conversely, suppose that  $I$  is  $n$ -semiprimary. Then  $x^n \in I$  for every  $x \in P$  by Theorem 2.3; so  $P^n = \{rx^n \mid r \in R, x \in P\} \subseteq I$  (cf. [12, Proposition 2.1 and Corollary 2.2]).

(b) This follows directly from part (a).

(c) If  $P = \sqrt{I}$  is not idempotent, then  $P^n \subseteq I$  for some positive integer  $n$  [23, Theorem 17.1(5)], and thus  $I$  is  $n$ -semiprimary by Theorem 2.5. For the ‘‘moreover’’ statement,  $P^n \subseteq I$  for some positive integer  $n$  since the prime ideal directly below  $P$  is  $Q = \bigcap_{n=1}^{\infty} P^n$  [23, Theorem 17.1(3)(4)].  $\square$

The following example illustrates the possible behavior of  $n$ -semiprimary ideals in valuation domains  $R$  with  $\dim(R) \leq 2$ . The details follow directly from Theorem 3.6 and well-known facts about the value group of a valuation domain (cf. [23, Chapter 3]). It is interesting to compare Theorem 3.6 (resp., Example 3.7) with [9, Theorem 5.5] (resp., [9, Example 5.6]) which concerns  $n$ -absorbing ideals in a valuation domain. There are  $n$ -semiprimary ideals that are not  $n$ -absorbing ideals in some valuation domains  $R$  since  $I$  is an  $n$ -semiprimary (resp.,  $n$ -absorbing) ideal of a valuation domain  $R$  if and only if  $P^n \subseteq I$  (resp.,  $P^n = I$ ).

**Example 3.7.** (a) Let  $R$  be a one-dimensional valuation domain with maximal ideal  $M$ . If  $M$  is principal, then  $R$  is a DVR, and thus every proper ideal of  $R$  is an  $n$ -semiprimary ideal for some positive integer  $n$ . If  $M$  is not principal, then  $M^2 = M$ , and hence  $\{0\}$  and  $M$  are the only proper ideals of  $R$  that are  $n$ -semiprimary for some positive integer  $n$ .

(b) Let  $R$  be a two-dimensional valuation domain with prime ideals  $\{0\} \subsetneq P \subsetneq M$  and value group  $G$ . If  $G = \mathbb{Z} \oplus \mathbb{Z}$  (all direct sums have the lexicographic order), then  $P^2 \neq P$  and  $M^2 \neq M$ ; so every proper ideal of  $R$  is  $n$ -semiprimary for some positive integer  $n$ . If  $G = \mathbb{Q} \oplus \mathbb{Q}$ , then  $P^2 = P$  and  $M^2 = M$ ; so  $\{0\}$ ,  $P$ , and  $M$  are the only ideals of  $R$  that are  $n$ -semiprimary for some positive integer  $n$ . If  $G = \mathbb{Z} \oplus \mathbb{Q}$ , then  $P^2 \neq P$  and  $M^2 = M$ ; so  $M$  and every ideal of  $R$  contained in  $P$  is  $n$ -semiprimary for some positive integer  $n$ , but no ideal properly between  $P$  and  $M$  is  $n$ -semiprimary for any positive integer  $n$ . If  $G = \mathbb{Q} \oplus \mathbb{Z}$ , then  $P^2 = P$  and  $M^2 \neq M$ ; so every ideal of  $R$  between  $P$  and  $M$  is  $n$ -semiprimary for some positive integer  $n$ , but  $\{0\}$  and  $P$  are the only ideals of  $R$  contained in  $P$  that are  $n$ -semiprimary for some positive integer  $n$ .

We end this section with two results on idealization. Let  $M$  be an  $R$ -module over a commutative ring  $R$ . The *idealization* of  $M$  is the commutative ring  $R(+M) = R \times M$  with addition and multiplication defined by  $(a, m) + (b, n) = (a + b, m + n)$  and  $(a, m)(b, n) = (ab, bm + an)$ , respectively, and identity  $(1, 0)$  (cf. [2], [26, Section 25]). Note that  $(\{0\}(+M))^2 = \{0\}$ ; so  $\{0\}(+M) \subseteq \text{nil}(R(+M))$ .

**Theorem 3.8.** *Let  $I$  be a proper ideal of a commutative ring  $R$ ,  $M$  an  $R$ -module, and  $S = IM$  a submodule of  $M$ . If  $I$  is an  $n$ -semiprimary ideal of  $R$ , then  $I(+S)$  is an  $(n + 1)$ -semiprimary ideal of  $R(+M)$ . Moreover, if  $I(+S)$  is an  $n$ -semiprimary ideal of  $R(+M)$ , then  $I$  is an  $n$ -semiprimary ideal of  $R$ .*

*Proof.* Let  $I$  be an  $n$ -semiprimary ideal of  $R$  and  $(a, m)^{n+1}(b, h)^{n+1} = (a^{n+1}b^{n+1}, z) \in I(+S)$  for  $(a, m), (b, h) \in R(+M)$ . Then  $a^n \in I$  or  $b^n \in I$  by Theorem 2.14(a) since  $I$  is an  $n$ -semiprimary ideal of  $R$ . We may assume that  $a^n \in I$ ; so  $(n + 1)a^n m \in IM = S$ . Thus  $(a, m)^{n+1} = (a^{n+1}, (n + 1)a^n m) \in I(+S)$ ; so  $I(+S)$  is an  $(n + 1)$ -semiprimary ideal of  $R(+M)$ . The “moreover” statement is clear.  $\square$

**Theorem 3.9.** *Let  $I$  a proper ideal of a commutative ring  $R$  with  $\text{char}(R) = n \geq 2$ ,  $M$  an  $R$ -module, and  $S$  a submodule of  $M$ . Then  $I(+S)$  is an  $n$ -semiprimary ideal of  $R(+M)$  if and only if  $I$  is an  $n$ -semiprimary ideal of  $R$ .*

*Proof.* If  $J = I(+S)$  is an  $n$ -semiprimary ideal of  $A = R(+M)$ , then clearly  $I$  is an  $n$ -semiprimary ideal of  $R$ . Conversely, assume that  $I$  is an  $n$ -semiprimary ideal of  $R$ . Let  $(a, m)^n(b, h)^n = (a^n b^n, z) \in J$  for  $(a, m), (b, h) \in A$ . Then  $a^n \in I$  or  $b^n \in I$  since  $I$  is an  $n$ -semiprimary ideal of  $R$ ; assume that  $a^n \in I$ . Since  $\text{char}(R) = n \geq 2$ ,

we have  $na^{n-1}m = 0 \in S$ . Thus  $(a, m)^n = (a^n, na^{n-1}m) = (a^n, 0) \in J$ ; so  $J$  is an  $n$ -semiprimary ideal of  $A$ .  $\square$

#### 4. $n$ -POWERFUL SEMIPRIMARY IDEALS AND $n$ -PVDs

In this section, we study  $n$ -powerful semiprimary ideals in integral domains and two generalizations of valuation domains, namely,  $n$ -pseudo-valuation domains ( $n$ -PVDs) and  $n$ -valuation domains ( $n$ -VDs).

Recall [17] (resp., [24]) that a proper ideal  $I$  of an integral domain  $R$  with quotient field  $K$  is *powerful* (resp., *strongly prime*) if whenever  $xy \in I$  for  $x, y \in K$ , then  $x \in R$  or  $y \in R$  (resp.,  $x \in I$  or  $y \in I$ ). We begin with an “ $n$ ” generalization.

**Definition 4.1.** Let  $R$  be an integral domain with quotient field  $K$  and  $n$  a positive integer. A proper ideal  $I$  of  $R$  is an  *$n$ -powerful ideal* of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in K$ , then  $x^n \in R$  or  $y^n \in R$ ; and  $I$  is an  *$n$ -powerful semiprimary ideal* of  $R$  if whenever  $x^n y^n \in I$  for  $x, y \in K$ , then  $x^n \in I$  or  $y^n \in I$ .

Thus a 1-powerful (resp., 1-powerful semiprimary) ideal is just a powerful (resp., strongly prime) ideal, and an  $n$ -powerful (resp.,  $n$ -powerful semiprimary) ideal is also an  $mn$ -powerful (resp.,  $mn$ -powerful semiprimary) ideal for every positive integer  $m$ . It is well known that prime ideals in a valuation domain are strongly prime ideals. From this observation, it easily follows that  $n$ -semiprimary ideals in a valuation domain are also  $n$ -powerful semiprimary ideals; so Theorem 3.6 and Example 3.7 also hold for  $n$ -powerful semiprimary ideals. However, an  $n$ -semiprimary ideal need not be an  $n$ -powerful semiprimary ideal. For example, let  $R = \mathbb{Z}_2[[X^2, X^3]]$ . Then its maximal ideal  $M = (X^2, X^3)$  is a prime (1-semiprimary) ideal, but not a strongly prime (1-powerful semiprimary) ideal. Also, see Example 4.5 for a 2-semiprimary ideal that is not 2-powerful semiprimary.

We next give a stronger result.

**Theorem 4.2.** *Let  $R$  be an integral domain with quotient field  $K$ .*

- (a) *Let  $I$  be an  $n$ -semiprimary ideal of  $R$ . If  $\sqrt{I}$  is a strongly prime ideal of  $R$ , then  $I$  is an  $n$ -powerful semiprimary ideal of  $R$ .*
- (b) *Let  $I \subseteq J$  be proper ideals of  $R$ . If  $J$  is an  $n$ -powerful ideal of  $R$ , then  $I$  is an  $n$ -powerful ideal of  $R$ .*
- (c) *Let  $I$  be an  $n$ -powerful (resp.,  $n$ -powerful semiprimary) ideal of  $R$  and  $S$  a multiplicatively closed subset of  $R$  with  $I \cap S = \emptyset$ . Then  $I_S$  is an  $n$ -powerful (resp.,  $n$ -powerful semiprimary) ideal of  $R_S$ .*

*Proof.* (a) Let  $P = \sqrt{I}$  and  $x^n y^n \in I \subseteq P$  for  $x, y \in K$ . Then  $x \in P$  or  $y \in P$  since  $P$  is a strongly prime ideal of  $R$ . Thus  $x^n \in I$  or  $y^n \in I$  by Theorem 2.3; so  $I$  is an  $n$ -powerful semiprimary ideal of  $R$ .

(b) Let  $x^n y^n \in I \subseteq J$  for  $x, y \in K$ . Then  $x^n \in R$  or  $y^n \in R$  since  $J$  is an  $n$ -powerful ideal of  $R$ . Thus  $I$  is an  $n$ -powerful ideal of  $R$ .

(c) This follows easily from the definitions.  $\square$

Note that every ideal in a valuation domain is powerful; so an  $n$ -powerful ideal need not be  $n$ -powerful semiprimary. However, for prime ideals, these two concepts coincide.

**Theorem 4.3.** *Let  $I$  be a prime ideal of an integral domain  $R$  with quotient field  $K$ . Then  $I$  is an  $n$ -powerful semiprimary ideal of  $R$  if and only if  $I$  is an  $n$ -powerful ideal of  $R$ .*

*Proof.* If  $I$  is an  $n$ -powerful semiprimary ideal of  $R$ , then  $I$  is certainly an  $n$ -powerful ideal. Conversely, assume that  $I$  is an  $n$ -powerful prime ideal of  $R$ . Let  $x^n y^n \in I$  for  $x, y \in K$ . First, suppose that  $x^n, y^n \in R$ . Since  $I$  is a prime ideal of  $R$ , then  $x^n \in I$  or  $y^n \in I$ . Thus we may assume that  $x^n \notin R$ , and hence  $y^n \in R$  since  $I$  is an  $n$ -powerful ideal of  $R$ . Since  $x^n \notin R$  and  $I$  is an  $n$ -powerful ideal of  $R$ , we have  $x^{2n} = x^n x^n \notin I$ . Assume that  $x^{2n} \in R$ ; so  $y^{2n}, x^{2n} \in R$ . Since  $x^{2n} y^{2n} \in I$  and  $x^{2n} \notin I$ , we have  $y^{2n} \in I$ . Since  $y^n \in R$ ,  $I$  is a prime ideal of  $R$ , and  $y^n y^n = y^{2n} \in I$ , we have  $y^n \in I$ . Now, assume that  $x^{2n} \notin R$ . Since  $(y^2/x^n)^n (x^2)^n = (y^{2n}/x^n y^n) x^{2n} = x^n y^n \in I$ ,  $x^{2n} \notin R$ , and  $I$  is an  $n$ -powerful ideal of  $R$ , we have  $y^{2n}/x^n y^n \in R$ . Thus  $y^{2n} = x^n y^n (y^{2n}/x^n y^n) \in I$ . Since  $y^n \in R$ ,  $I$  is a prime ideal of  $R$ , and  $y^{2n} = y^n y^n \in I$ , we have  $y^n \in I$ . Hence  $I$  is an  $n$ -powerful semiprimary ideal of  $R$ .  $\square$

**Theorem 4.4.** *Let  $P \subseteq Q$  be prime ideals of an integral domain  $R$ . If  $Q$  is an  $n$ -powerful semiprimary ideal of  $R$ , then  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ .*

*Proof.* Let  $Q$  be an  $n$ -powerful ideal of  $R$ ; so  $P$  is an  $n$ -powerful ideal of  $R$  by Theorem 4.2(b). Thus  $P$  is an  $n$ -powerful semiprimary ideal of  $R$  by Theorem 4.3.  $\square$

Let  $I$  be a proper ideal of an integral domain  $R$ . As the “powerful” analogs of  $W_R(I)$  and  $\delta_R(I)$ , we define  $\overline{W}_R(I) = \{n \in \mathbb{N} \mid I \text{ is an } n\text{-powerful semiprimary ideal of } R\}$  and  $\overline{\delta}_R(I) = \min \overline{W}_R(I)$  (let  $\overline{\delta}_R(I) = \infty$  if  $\overline{W}_R(I) = \emptyset$ ). Note that  $\overline{W}_R(I) \subseteq W_R(I)$  and  $\delta_R(I) \leq \overline{\delta}_R(I)$ . The next example shows that the analogs of Theorem 2.14(b) and Theorem 4.2(b) do not hold for  $n$ -powerful semiprimary ideals. In particular, if  $I$  is an  $n$ -powerful semiprimary ideal, then  $I$  is an  $n$ -semiprimary ideal. Thus  $I$  is also an  $m$ -semiprimary ideal for every integer  $m \geq n$ , but  $I$  need not be an  $m$ -powerful semiprimary ideal.

**Example 4.5.** Let  $R = F[[X^2, X^5]] = F + FX^2 + X^4F[[X]]$ , where  $F$  is a field. Then  $R$  is quasilocal with maximal ideal  $M = (X^2, X^5) = FX^2 + X^4F[[X]]$  and quotient field  $K = F[[X]][1/X]$ . Clearly  $M$  is a 2-semiprimary ideal of  $R$ , but not a 3-powerful semiprimary ideal of  $R$  since  $X^3 X^3 = X^6 \in M$ , but  $X^3 \notin M$ . Moreover,  $M$  is a 2-powerful semiprimary ideal of  $R$  if and only if  $\text{char}(F) = 2$ , and  $M$  is an  $n$ -powerful semiprimary ideal of  $R$  for every integer  $n \geq 4$ . So, for  $R = \mathbb{Z}_2[[X^2, X^5]]$ ,  $M$  is a 2-powerful semiprimary ideal, but not a 3-powerful semiprimary ideal, and  $\overline{W}_R(M) = \mathbb{N} \setminus \{1, 3\}$ . Thus the “powerful” analog of Theorem 2.14(b) fails for  $M$ . Let  $I = X^4F[[X]]$ . Then  $I$  is a 2-semiprimary ideal of  $R$ , but not a 2-powerful semiprimary ideal of  $R$  since  $X^2 X^2 \in I$ , but  $X^2 \notin I$ . So the “semiprimary” analog of Theorem 4.2(b) fails for  $I \subseteq J = M$  when  $\text{char}(F) = 2$ .

Recall [24] that an integral domain  $R$  is a *pseudo-valuation domain* (PVD) if every prime ideal of  $R$  is strongly prime. A PVD is necessarily quasilocal [24, Corollary 1.3]. A quasilocal integral domain  $R$  with maximal ideal  $M$  is a PVD  $\Leftrightarrow M$  is strongly prime [24, Theorem 1.4], and  $R$  is a PVD  $\Leftrightarrow (M : M)$  is a valuation domain with maximal ideal  $M$  [11, Proposition 2.5]. Let  $T = K + M$  be a valuation domain, where  $K$  is a field and  $M$  is the maximal ideal of  $T$ . Then for a proper subfield  $k$  of  $K$ , the subring  $R = k + M$  is a PVD which is not a valuation domain [24, Example 2.1]. By Theorem 4.2(a), every  $n$ -semiprimary ideal in a PVD is an  $n$ -powerful semiprimary ideal.

We next give an “ $n$ ” generalization of PVDs.

**Definition 4.6.** Let  $R$  be an integral domain and  $n$  a positive integer. Then  $R$  is an  $n$ -pseudo-valuation domain ( $n$ -PVD) if every prime ideal of  $R$  is an  $n$ -powerful semiprimary ideal of  $R$ .

Note that a 1-PVD is just a PVD and an  $n$ -PVD is also an  $mn$ -PVD for every positive integer  $m$ . The next several results show that  $n$ -PVDs behave very much like PVDs (cf. [1], [6], [8], [11], [17], [24], and [25]).

**Theorem 4.7.** *Let  $R$  be an  $n$ -PVD. Then  $R$  is quasilocal.*

*Proof.* By way of contradiction, assume that  $M$  and  $N$  are distinct maximal ideals of  $R$ . Let  $x \in M \setminus N$  and  $y \in N \setminus M$ . Then  $(x/y)^n(y^2)^n = (x^n/y^n)y^{2n} = x^n y^n \in M$ , and thus  $(x/y)^n \in M$  since  $M$  is an  $n$ -powerful semiprimary ideal of  $R$  and  $(y^2)^n \notin M$ . Hence  $x^n = (x/y)^n y^n \in N$ ; so  $x \in N$ , a contradiction. Thus  $R$  is quasilocal.  $\square$

In view of Theorem 4.3, Theorem 4.4, and the proof of Theorem 4.7, we have the following result.

**Corollary 4.8.** *An integral domain  $R$  is an  $n$ -PVD if and only if some maximal ideal of  $R$  is an  $n$ -powerful semiprimary ideal of  $R$ , if and only if some maximal ideal of  $R$  is an  $n$ -powerful ideal of  $R$ .*

Recall ([20], [14]) that a prime ideal  $P$  of a commutative ring  $R$  is a *divided prime ideal* of  $R$  if  $x|p$  (in  $R$ ) for every  $x \in R \setminus P$  and  $p \in P$  (i.e.,  $(x)$  is comparable to  $P$  for every  $x \in R$ ), and  $R$  is a *divided ring* if every prime ideal of  $R$  is divided. We next give the “ $n$ ” generalization.

**Definition 4.9.** Let  $R$  be a commutative ring and  $n$  a positive integer. Then a prime ideal  $P$  of  $R$  is an  $n$ -divided prime ideal of  $R$  if  $x^n|p^n$  (in  $R$ ) for every  $x \in R \setminus P$  and  $p \in P$ . Moreover,  $R$  is an  $n$ -divided ring if every prime ideal of  $R$  is an  $n$ -divided prime ideal of  $R$ .

A 1-divided prime ideal (resp., ring) is just a divided prime ideal (resp., ring), and an  $n$ -divided prime ideal is  $mn$ -divided for every positive integer  $m$ . Thus an  $n$ -divided ring is  $mn$ -divided for every positive integer  $m$ .

The next several results show that  $n$ -divided rings behave very much like divided rings (cf. [14], [20]).

**Theorem 4.10.** *Let  $R$  be an  $n$ -divided commutative ring. Then the set of prime ideals of  $R$  is linearly ordered by inclusion. In particular,  $R$  is quasilocal.*

*Proof.* Let  $P$  and  $Q$  be prime ideals of an  $n$ -divided commutative ring  $R$  with  $P \not\subseteq Q$ . We show that  $Q \subseteq P$ . Let  $x \in P \setminus Q$ ; then  $x^n|q^n$  for every  $q \in Q$  since  $Q$  is an  $n$ -divided prime ideal of  $R$ . Thus  $q^n \in (x^n) \subseteq P$ ; so  $q \in P$  for every  $q \in Q$ . Hence  $Q \subseteq P$ .  $\square$

**Theorem 4.11.** *Let  $P$  a prime ideal of an integral domain  $R$ . If  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ , then  $P$  is an  $n$ -divided prime ideal of  $R$ . Moreover, the set of prime ideals of  $R$  that are contained in  $P$  is linearly ordered by inclusion.*

*Proof.* Let  $x \in R \setminus P$  and  $p \in P$ . Then  $(p/x)^n x^n = (p^n/x^n)x^n = p^n \in P$ . Thus  $p^n/x^n \in P$  since  $x^n \notin P$  and  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ . Hence

$p^n = (p^n/x^n)x^n$ , so  $x^n \mid p^n$  (in  $R$ ). Thus  $P$  is an  $n$ -divided prime ideal of  $R$ . Now suppose that  $F$  and  $H$  are distinct prime ideals of  $R$  contained in  $P$ . Then  $F$  and  $H$  are  $n$ -powerful semiprimary ideals of  $R$  by Theorem 4.4, and hence are  $n$ -divided prime ideals. The proof of Theorem 4.10 shows that  $F$  and  $H$  are comparable under inclusion.  $\square$

**Corollary 4.12.** *Let  $R$  be an  $n$ -PVD. Then  $R$  is an  $n$ -divided domain and the set of prime ideals of  $R$  is linearly ordered by inclusion. Moreover, if  $R$  is Noetherian, then  $\dim(R) \leq 1$ .*

*Proof.* We need only prove the ‘‘moreover’’ statement; it follows directly from [27, Theorem 144].  $\square$

Let  $R$  be an integral domain with quotient field  $K$ ,  $S \subseteq R$ , and  $n$  a positive integer. Define  $E_n(S) = \{x \mid x^n \notin S, x \in K\}$  and  $A_n(S) = \{x^n \mid x^n \in S, x \in K\}$ . We next use these two sets to give another characterization of  $n$ -powerful semiprimary ideals. Note that actually  $x^{-n}d \in A_n(P)$  in Theorem 4.13 and Corollary 4.15(4), and  $x^{-n}d \in A_n(M)$  in Corollary 4.16(3).

**Theorem 4.13.** *Let  $P$  a prime ideal of an integral domain  $R$  with quotient field  $K$ . Then  $P$  is an  $n$ -powerful semiprimary ideal of  $R$  if and only if  $x^{-n}d \in P$  for every  $x \in E_n(P)$  and  $d \in A_n(P)$ .*

*Proof.* Suppose that  $x^{-n}d \in P$  for every  $x \in E_n(P)$  and  $d \in A_n(P)$ . Let  $x^ny^n \in P$  for  $x, y \in K$  with  $x^n \notin P$ ; so  $x \in E_n(P)$ . Since  $x^ny^n = (xy)^n \in A_n(P)$ , we have  $y^n = x^{-n}(x^ny^n) \in P$ . Thus  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ .

Conversely, suppose that  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ . Let  $d \in A_n(P)$ ; so  $d = a^n \in P$  for some  $a \in K$  and  $x^n(x^{-1}a)^n = x^nx^{-n}a^n = a^n \in P$  for every  $0 \neq x \in K$ . Suppose that  $x \in E_n(P)$ . Then  $x^n \notin P$ ; so  $(x^{-1}a)^n \in P$  since  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ . Thus  $x^{-n}d = x^{-n}a^n = (x^{-1}a)^n \in P$ .  $\square$

The proof of the following result is similar to that of Theorem 4.13, and thus will be omitted.

**Theorem 4.14.** *Let  $I$  a proper ideal of an integral domain  $R$ . Then  $I$  is an  $n$ -powerful ideal of  $R$  if and only if  $x^{-n}d \in R$  for every  $x \in E_n(R)$  and  $d \in A_n(I)$ .*

In view of Theorem 4.3, Theorem 4.13, and Theorem 4.14, we have the following result.

**Corollary 4.15.** *Let  $P$  be a prime ideal of an integral domain  $R$ . Then the following statements are equivalent.*

- (1)  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ .
- (2)  $P$  is an  $n$ -powerful ideal of  $R$ .
- (3)  $x^{-n}d \in R$  for every  $x \in E_n(R)$  and  $d \in A_n(P)$ .
- (4)  $x^{-n}d \in P$  for every  $x \in E_n(P)$  and  $d \in A_n(P)$ .

In view of Corollary 4.8, Theorem 4.13, and Theorem 4.14, we have the following result.

**Corollary 4.16.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$ . Then the following statements are equivalent.*

- (1)  $R$  is an  $n$ -PVD.
- (2)  $x^{-n}d \in R$  for every  $x \in E_n(R)$  and  $d \in A_n(M)$ .

(3)  $x^{-n}d \in M$  for every  $x \in E_n(M)$  and  $d \in A_n(M)$ .

If  $R$  is a PVD, then  $R/P$  is also a PVD for  $P$  a prime ideal of  $R$  [21, Lemma 4.5(i)]. The analogous result holds for  $n$ -PVDs.

**Theorem 4.17.** *Let  $P$  be a prime ideal of an  $n$ -PVD  $R$ . Then  $R/P$  is an  $n$ -PVD.*

*Proof.* Let  $M$  be the maximal ideal of  $R$ ,  $K$  the quotient field of  $R$ ,  $F = R_P/PR_P$  the quotient field of  $A = R/P$ , and  $H_n(M/P) = \{x^n \in M/P \mid x \in F\}$ . Suppose that  $x = a + P, y = b + P \in A$ , and  $x^n \nmid y^n$  in  $A$ . Then  $a^n \nmid b^n$  in  $R$ ; so  $b^n \mid a^n d$  in  $R$  for every  $d \in A_n(M)$  by Corollary 4.16. Thus  $y^n \mid x^n h$  in  $A$  for every  $h \in H_n(M/P)$ ; so  $A$  is an  $n$ -PVD by Corollary 4.16 again.  $\square$

Let  $n$  be a positive integer. Recall that an integral domain  $R$  with quotient field  $K$  is  $n$ -root closed if whenever  $x^n \in R$  for  $x \in K$ , then  $x \in R$ ; and  $R$  is root closed if  $R$  is  $n$ -root closed for every positive integer  $n$ . For example, an integrally closed integral domain is root closed. Note that  $R$  is  $mn$ -root closed if and only if  $R$  is  $m$ -root closed and  $n$ -root closed. Thus  $\mathcal{C}(R) = \{n \in \mathbb{N} \mid R \text{ is } n\text{-root closed}\}$  is a multiplicative submonoid on  $\mathbb{N}$  generated by some set of prime numbers. Moreover, for  $S$  any multiplicative submonoid of  $\mathbb{N}$  generated by a set of prime numbers,  $S = \mathcal{C}(R)$  for some integral domain  $R$  [7, Theorem 2.7].

For  $n$ -root closed integral domains, the  $n$ -PVD and PVD concepts coincide.

**Theorem 4.18.** *Let  $R$  be an  $n$ -root closed integral domain with quotient field  $K$ . Then  $R$  is an  $n$ -PVD if and only if  $R$  is a PVD. In particular, an integrally closed  $n$ -PVD is a PVD.*

*Proof.* If  $R$  is a PVD, then clearly  $R$  is an  $n$ -PVD. Conversely, let  $R$  be an  $n$ -root closed  $n$ -PVD with maximal ideal  $M$ . We show that  $M$  is a powerful ideal of  $R$ . Let  $xy \in M$  for  $x, y \in K$  and  $x \notin R$ . Then  $x^n y^n \in M$  and  $x^n \notin R$  since  $R$  is  $n$ -root closed. Thus  $y^n \in M \subseteq R$  since  $M$  is an  $n$ -powerful semiprimary ideal of  $R$ , and hence  $y \in R$  since  $R$  is  $n$ -root closed. Thus  $M$  is a powerful ideal of  $R$ ; so  $M$  is a strongly prime ideal of  $R$  (i.e.,  $M$  is a 1-powerful semiprimary ideal of  $R$ ) by Theorem 4.3. Hence  $R$  is a PVD. The ‘‘in particular’’ statement is clear.  $\square$

Recall ([4], [3], [5], [29]) that an integral domain  $R$  with quotient field  $K$  is an *almost valuation domain* if for every  $0 \neq x \in K$ , there is a positive integer  $n$  (depending on  $x$ ) such that  $x^n \in R$  or  $x^{-n} \in R$ . We have the following ‘‘ $n$ ’’ generalization.

**Definition 4.19.** Let  $n$  be a positive integer. An integral domain  $R$  with quotient field  $K$  is an  $n$ -valuation domain ( $n$ -VD) if  $x^n \in R$  or  $x^{-n} \in R$  for every  $0 \neq x \in K$ .

It is clear that a valuation domain is an  $n$ -VD for every positive integer  $n$ , an  $n$ -root closed  $n$ -VD is a valuation domain, an  $n$ -VD is an almost valuation domain, an  $n$ -VD is also an  $mn$ -VD for every positive integer  $m$ , and an  $n$ -VD is an  $n$ -PVD. Moreover, an  $n$ -VD is quasilocal, an overring of an  $n$ -VD is also an  $n$ -VD, and a Noetherian  $n$ -VD has (Krull) dimension at most one.

We have the following elementary results about  $n$ -VDs which show that  $n$ -VDs behave very much like valuation domains (cf. [23, Chapter 3]). In [1, page 3], it was observed that  $R$  is a valuation domain if and only if  $R$  is a strongly prime ideal of  $R$  (here, and in Theorem 4.20(a)(5), we drop the usual assumption that a prime ideal is a proper ideal).

**Theorem 4.20.** *Let  $R$  be an integral domain with quotient field  $K$  and  $n$  a positive integer.*

(a) *The following statements are equivalent.*

- (1)  $R$  is an  $n$ -VD.
- (2)  $x^n \mid y^n$  or  $y^n \mid x^n$  for every  $0 \neq x, y \in K$ .
- (3)  $x^n \mid y^n$  or  $y^n \mid x^n$  for every  $0 \neq x, y \in R$ .
- (4) Let  $G$  be the group of divisibility of  $R$ . Then for every  $g \in G$ , either  $ng \geq 0$  or  $ng < 0$ .
- (5)  $R$  is an  $n$ -powerful semiprimary ideal of  $R$ .

(b) *Let  $R$  be an  $n$ -VD. Then  $R$  is an  $n$ -divided domain, and thus the prime ideals of  $R$  are linearly ordered by inclusion.*

(c) *Let  $R$  be an  $n$ -VD and  $x \in K$ . If  $x^n$  is integral over  $R$ , then  $x^n \in R$ .*

*Proof.* The proofs are essentially the same as for valuation domains. See [23, Theorem 16.3] for part (a) and [23, Theorem 17.5] for part (c). Part (b) follows from Corollary 4.12 since an  $n$ -VD is also an  $n$ -PVD.  $\square$

An  $n$ -VD is always an  $n$ -PVD, but an  $n$ -PVD need not be an  $n$ -VD. Also, an almost valuation domain need not be an  $n$ -VD for any positive integer  $n$ .

**Example 4.21.** (a) Let  $R = \mathbb{Q} + X\mathbb{R}[[X]]$ . Then  $R$  is a PVD with maximal ideal  $X\mathbb{R}[[X]]$  and quotient field  $\mathbb{R}[[X]][1/X]$ , and thus  $R$  is an  $n$ -PVD for every positive integer  $n$ . However,  $R$  is not an  $n$ -VD for any positive integer  $n$  since  $\pi^n, \pi^{-n} \notin R$  for every positive integer  $n$ .

(b) Let  $R = \mathbb{Z}_p + XF[[X]]$ , where  $p$  is a positive prime integer and  $F = \overline{\mathbb{Z}_p}$  is the algebraic closure of  $\mathbb{Z}_p$ . Then  $R$  is an almost valuation domain with maximal ideal  $XF[[X]]$  and quotient field  $F[[X]][1/X]$ , but not an  $n$ -VD for any positive integer  $n$ . This follows from the fact that for every  $0 \neq a \in F$ , there is a positive integer  $n$  such that  $a^n = 1$ ; but for every positive integer  $n$ , there is a  $b \in F$  such that  $b^n \notin \mathbb{Z}_p$  and  $b^{-n} \notin \mathbb{Z}_p$ . Note that  $R$  is also a PVD, and thus an  $n$ -PVD for every positive integer  $n$ .

In some cases, an overring of an  $n$ -PVD is also an  $n$ -VD.

**Theorem 4.22.** *Let  $R$  be an  $n$ -PVD with maximal ideal  $M$ , quotient field  $K$ , and  $V$  an overring of  $R$  such that  $1/s \in V$  for some  $0 \neq s \in M$ . Then  $V$  is an  $n$ -VD, and thus  $V$  is an almost valuation domain.*

*Proof.* Let  $x \in K$  with  $x^n \notin V$ ; so  $x \in E_n(R)$ . Then  $x^{-n}d \in M$  for every  $d \in A_n(M)$  by Corollary 4.16. In particular,  $a = x^{-n}s^n \in M$  since  $d = s^n \in A_n(M)$ , and thus  $x^{-n} = a/s^n \in V$  since  $1/s \in V$ . Hence  $V$  is an  $n$ -VD, and thus  $V$  is an almost valuation domain.  $\square$

By Theorem 4.2(c), if  $R$  is an  $n$ -PVD, then  $R_P$  is also an  $n$ -PVD for every nonmaximal prime ideal  $P$  of  $R$ . We next give a stronger result;  $R_P$  is an  $n$ -VD.

**Theorem 4.23.** *Let  $R$  be an  $n$ -PVD with maximal ideal  $M$  and  $P \subsetneq M$  a prime ideal of  $R$ . Then  $R_P$  is an  $n$ -VD, and thus  $R_P$  is an almost valuation domain. Moreover,  $x^n \in R$  for every  $x \in P_P$ , and hence  $P_P \subsetneq \overline{R}$ .*

*Proof.* Since  $P \subsetneq M$ , there is an  $s \in M \setminus P$ . Thus  $1/s \in R_P$ ; so  $R_P$  is an  $n$ -VD (and hence also an almost valuation domain) by Theorem 4.22. Let  $x \in P_P$ ; so  $x = a/s$  for some  $a \in P$  and  $s \in R \setminus P$ . Thus  $s^n \mid a^n$  (in  $R$ ) since  $P$  is an  $n$ -divided prime ideal of  $R$  by Theorem 4.11. Hence  $x^n = a^n/s^n \in R$ ; so  $P_P \subsetneq \overline{R}$ .  $\square$

We next show that  $n$ -divided principal prime ideals are actually maximal ideals.

**Theorem 4.24.** *Let  $R$  be an integral domain  $R$  with quotient field  $K$  and (nonzero) principal prime ideal  $P$ . If  $P$  is an  $n$ -divided ideal of  $R$ , then  $P$  is a maximal ideal of  $R$ . Moreover, if  $P$  is also an  $n$ -powerful semiprimary ideal of  $R$ , then  $P$  is a maximal ideal of  $R$  and  $R$  is an  $n$ -VD.*

*Proof.* Let  $P = (p)$  for a prime element  $p$  of  $R$ . By way of contradiction, assume that  $P$  is not a maximal ideal; so there is a nonunit  $x \in R \setminus P$ . If  $P$  is an  $n$ -divided prime ideal of  $R$ , then there is a  $y \in R$  with  $p^n = x^n y p^n$  or  $p^n = x^n w p^m$  for some positive integer  $m < n$  and  $w \in R \setminus P$ . If  $p^n = x^n y p^n$ , then  $1 = x^n y$ , and thus  $x \in U(R)$ , a contradiction. If  $p^n = x^n w p^m$ , then  $x^n w = p^{n-m} \in P$ , which is a contradiction since  $x \notin P$  and  $w \notin P$ . Hence  $P$  is a maximal ideal of  $R$ .

Now, suppose that  $P = (p)$  is an  $n$ -powerful semiprimary ideal of  $R$ . Then  $P$  is an  $n$ -divided prime ideal of  $R$  by Theorem 4.11. Thus  $P$  is a maximal ideal of  $R$ , and hence  $R$  is an  $n$ -PVD by Corollary 4.8. Finally, we show that  $R$  is an  $n$ -VD. Let  $x \in K$ , and suppose that  $x^n \notin R$ . Then  $x^n \notin P$ , and thus  $x^{-n} p^n \in P$  by Theorem 4.13. Since  $x^{-n} p^n \in P = (p)$ , we have  $x^{-n} p^n = h p^n$  for some  $h \in R$  or  $x^{-n} p^n = d p^m$  for some positive integer  $m < n$  and  $d \in U(R)$ . If  $x^{-n} p^n = d p^m$ , then  $x^n = d^{-1} p^{n-m} \in R$ , a contradiction. Thus  $x^{-n} p^n = h p^n$  for some  $h \in R$ , and hence  $x^{-n} = h \in R$ . Thus  $R$  is an  $n$ -VD.  $\square$

We have already observed several parts of the next theorem. One interesting consequence is that if  $P$  is an  $n$ -powerful semiprimary prime ideal of an integral domain  $R$  with quotient field  $K$ , then  $\{x \in K \mid x^m \in P \text{ for some positive integer } m\} = \{x \in K \mid x^n \in P\}$  (cf. Theorem 2.3).

**Theorem 4.25.** *Let  $P$  be a prime ideal of an integral domain  $R$  with quotient field  $K$ . If  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ , then  $P$  is an  $mn$ -powerful semiprimary ideal of  $R$  for every positive integer  $m$ . Furthermore, if  $x^m \in P$  for a positive integer  $m$  and  $x \in K$ , then  $x^n \in P$ . In particular, if  $R$  is an  $n$ -PVD, then  $R$  is an  $mn$ -PVD for every positive integer  $m$ .*

*Proof.* Let  $m$  be a positive integer. Assume that  $x^{mn} y^{mn} \in P$  for  $x, y \in K$ . Then  $(x^m)^n (y^m)^n \in P$ . Since  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ ,  $(x^m)^n = x^{mn} \in P$  or  $(y^m)^n = y^{mn} \in P$ . Thus  $P$  is an  $mn$ -powerful semiprimary ideal of  $R$ . Next, assume that  $x^m \in P$  for  $x \in K$  and some positive integer  $m$ ; so  $x^{mn} = (x^m)^n \in P$ . Let  $d$  be the least positive integer such that  $x^{dn} \in P$ . Since  $(x^{d-1})^n x^n = x^{dn} \in P$  and  $P$  is an  $n$ -powerful semiprimary ideal of  $R$ , we have  $(x^{d-1})^n \in P$  or  $x^n \in P$ . Hence  $d = 1$ , and thus  $x^n \in P$ . The ‘‘in particular’’ statement is clear.  $\square$

The next several results concern integral overrings of an  $n$ -PVD. In particular, an integral overring of an  $n$ -PVD is an  $n$ -PVD, and the integral closure of an  $n$ -PVD is a PVD. Note that  $\{x \in K \mid x^n \in M\} = \{x \in \overline{R} \mid x^n \in M\}$  in the next several results and  $\sqrt{MB} = \sqrt{M\overline{R}} \cap B$  for  $B$  an integral overring of  $R$ .

**Theorem 4.26.** *Let  $R$  be an  $n$ -PVD with maximal ideal  $M$  and quotient field  $K$ . If  $B$  is an integral overring of  $R$ , then  $B$  is an  $n$ -PVD with maximal ideal  $M_B = \sqrt{MB} = \{x \in B \mid x^n \in M\}$ .*

*Proof.* Let  $m \in M$ . Then  $\sqrt{mR}$  is a prime ideal of  $R$  since the prime ideals of  $R$  are linearly ordered (under inclusion) by Corollary 4.12, and thus  $\sqrt{mR}$  is an

$n$ -powerful semiprimary ideal of  $R$  since  $R$  is an  $n$ -PVD. We show that  $\sqrt{mB}$  is an  $n$ -powerful semiprimary ideal of  $B$  and  $\sqrt{mB} = \{x \in B \mid x^n \in \sqrt{mR}\}$ . Let  $x^n y^n \in \sqrt{mB}$  for  $0 \neq x, y \in K$ . Then  $x^{nk} y^{nk} = (xy)^{nk} = fm$  for some positive integer  $k$  and  $0 \neq f \in B$ . Note that  $f^{-n} \notin M$ ; for if  $f^{-n} \in M$ , then  $1/a = f^n \in B$  for some  $a \in M$ , a contradiction since  $B$  is integral over  $R$ . Then  $f^n m^n \in \sqrt{mR}$  since  $f^{-n}(fm)^n = m^n \in \sqrt{mR}$ ,  $f^{-n} \notin \sqrt{mR} \subseteq M$ , and  $\sqrt{mR}$  is an  $n$ -powerful semiprimary ideal of  $R$ . Thus  $(x^{nk})^n (y^{nk})^n = (xy)^{nkn} = f^n m^n \in \sqrt{mR}$ ; so  $x^{nkn} \in \sqrt{mR} \subseteq \sqrt{mB}$  or  $y^{nkn} \in \sqrt{mR} \subseteq \sqrt{mB}$ . Hence  $x^n \in \sqrt{mR} \subseteq \sqrt{mB}$  or  $y^n \in \sqrt{mR} \subseteq \sqrt{mB}$  by Theorem 4.25. Thus  $\sqrt{mB}$  is an  $n$ -powerful semiprimary ideal of  $B$ , and hence a prime ideal of  $B$  by Theorem 2.3. A slight modification of the above proof also shows that  $\sqrt{mB} = \{x \in B \mid x^n \in \sqrt{mR}\}$ .

We next show that  $M_B = \{x \in B \mid x^n \in M\}$  is an  $n$ -powerful semiprimary ideal of  $B$ . First, we show that  $M_B$  is an ideal of  $B$ . Let  $x_1, x_2 \in M_B$ ; so  $x_1^n = m_1 \in M$  and  $x_2^n = m_2 \in M$ . Thus  $x_1 \in \sqrt{m_1 B}$  and  $x_2 \in \sqrt{m_2 B}$ . Since the prime ideals of  $R$  are linearly ordered, we may assume that  $\sqrt{m_1 R} \subseteq \sqrt{m_2 R}$ , and hence  $\sqrt{m_1 B} \subseteq \sqrt{m_2 B}$ . Thus  $x_1 + x_2 \in \sqrt{m_2 B} = \{x \in B \mid x^n \in \sqrt{m_2 R}\} \subseteq M_B$ . Next, let  $x \in M_B$  and  $y \in B$ . Then  $x^n = m_3 \in M$ ; so  $x \in \sqrt{m_3 B}$ . Thus  $xy \in \sqrt{m_3 B} \subseteq M_B$ ; so  $M_B$  is an ideal of  $B$ . A similar argument to that for  $\sqrt{mB}$  above shows that if  $x^n y^n \in M_B$  for  $0 \neq x, y \in K$ , then  $x^n \in \sqrt{mR} \subseteq M \subseteq M_B$  or  $y^n \in \sqrt{mR} \subseteq M \subseteq M_B$ . Hence  $M_B$  is an  $n$ -powerful semiprimary ideal of  $B$ , and thus  $M_B$  is a prime ideal of  $B$  since it is a radical ideal of  $B$  by Theorem 4.25. Hence  $M_B$  is a maximal ideal of  $B$  since  $B$  is integral over  $R$  and  $M_B \cap R = M$ ; so  $B$  is an  $n$ -PVD by Corollary 4.8. Clearly  $M_B = \{x \in B \mid x^n \in M\} \subseteq \sqrt{MB}$ , and  $\sqrt{MB} \subseteq M_B$  since  $MB \subsetneq B$  as  $B$  is integral over  $R$ . Thus  $M_B = \sqrt{MB}$ .  $\square$

**Corollary 4.27.** *Let  $R$  be an  $n$ -PVD with maximal ideal  $M$  and quotient field  $K$ . Then  $\bar{R}$  is a PVD (1-PVD) with maximal ideal  $\sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$ .*

*Proof.* By Theorem 4.26,  $\bar{R}$  is an  $n$ -PVD with maximal ideal  $\sqrt{M\bar{R}} = M_{\bar{R}} = \{x \in \bar{R} \mid x^n \in M\} = \{x \in K \mid x^n \in M\}$ . Thus  $\bar{R}$  is a PVD by Theorem 4.18.  $\square$

**Corollary 4.28.** *Let  $P$  be a nonzero finitely generated prime ideal of an  $n$ -PVD  $R$ . Then  $W = (P : P)$  is an  $n$ -PVD with maximal ideal  $\sqrt{MW} = \{x \in W \mid x^n \in M\}$ . In particular, if  $R$  is a Noetherian  $n$ -PVD with maximal ideal  $M$ , then  $(M : M)$  is an  $n$ -PVD.*

*Proof.* Note that  $W = (P : P)$  is integral over  $R$  since  $P$  is finitely generated. Thus  $W$  is an  $n$ -PVD with maximal ideal  $\sqrt{MW} = \{x \in W \mid x^n \in M\}$  by Theorem 4.26. The ‘‘in particular’’ statement is clear. (However, recall that a Noetherian  $n$ -PVD  $R$  has  $\dim(R) \leq 1$  by Corollary 4.12).  $\square$

The converse of Corollary 4.27 also holds.

**Theorem 4.29.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ . Then  $R$  is an  $n$ -PVD if and only if  $\bar{R}$  is a PVD with maximal ideal  $\sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$ .*

*Proof.* Let  $R$  be an  $n$ -PVD. Then  $\bar{R}$  is a PVD with maximal ideal  $\sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$  by Corollary 4.27. Conversely, suppose that  $\bar{R}$  is a PVD with maximal ideal  $N = \sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$ . Then  $M = R \cap N$  since  $M \subseteq N$ . Let  $x^n y^n = (xy)^n \in M$  for  $x, y \in K$ ; so  $xy \in N$ . Thus  $x \in N$  or  $y \in N$  since  $N$  is

a strongly prime ideal of  $\bar{R}$ . Hence  $x^n \in M$  or  $y^n \in M$ ; so  $M$  is an  $n$ -powerful semiprimary ideal of  $R$ . Thus  $R$  is an  $n$ -PVD by Corollary 4.8.  $\square$

**Corollary 4.30.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ . Then the following statements are equivalent.*

- (1)  $R$  is an  $n$ -PVD.
- (2)  $\bar{R}$  is a PVD with maximal ideal  $\sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$ .
- (3)  $N = \sqrt{M\bar{R}} = \{x \in K \mid x^n \in M\}$  is a maximal ideal of  $\bar{R}$  such that  $(N : N)$  is a valuation domain with maximal ideal  $N$ .

*Proof.* (1)  $\Leftrightarrow$  (2) is Theorem 4.29, and (2)  $\Leftrightarrow$  (3) is clear by [11, Proposition 2.5].  $\square$

We have seen that integral overrings of an  $n$ -PVD are also  $n$ -PVDs. We next determine when every overring of an  $n$ -PVD is an  $n$ -PVD. Note that an integrally closed PVD need not be a valuation domain. For example,  $R = \mathbb{Q} + XC[[X]]$  is a PVD, and  $\bar{R} = \bar{\mathbb{Q}} + XC[[X]]$  is a PVD, but not a valuation domain, where  $\bar{\mathbb{Q}}$  is the algebraic closure of  $\mathbb{Q}$ . In this case,  $\mathbb{Q}[\pi] + XC[[X]]$  is a (non-integral) overring of  $R$  which is not an  $n$ -VD or  $n$ -PVD for any positive integer  $n$ .

**Theorem 4.31.** *Let  $R$  be an  $n$ -PVD with maximal ideal  $M$ . Then every overring of  $R$  is an  $n$ -PVD if and only if  $\bar{R}$  is a valuation domain. Moreover, if  $\bar{R}$  is a valuation domain, then every non-integral overring of  $R$  is an  $n$ -VD.*

*Proof.* Suppose that every overring of  $R$  is an  $n$ -PVD. Since  $\bar{R}$  is a PVD by Theorem 4.18, the proof of [25, Proposition 2.7] shows that if  $\bar{R}$  is not a valuation domain, then there is a non-quasilocal overring  $B$  of  $\bar{R}$  (and hence  $B$  is an overring of  $R$ ). Thus  $B$  cannot be an  $n$ -PVD by Theorem 4.7; so  $\bar{R}$  is a valuation domain.

Conversely, suppose that  $\bar{R}$  is a valuation domain with maximal ideal  $N$ . Let  $B$  be an overring of  $R$ . If  $B$  is integral over  $R$ , then  $B$  is an  $n$ -PVD by Theorem 4.26; so assume that  $B$  is not integral over  $R$ . Let  $b \in B \setminus \bar{R}$ . Then  $b^{-1} \in N$  since  $\bar{R}$  is a valuation domain; so  $m = b^{-n} = (b^{-1})^n \in M$  by Corollary 4.27 since  $\bar{R}$  is a valuation domain (and thus a PVD). Hence  $1/m = b^n \in B$ ; so  $B$  is an  $n$ -VD, and thus an  $n$ -PVD, by Theorem 4.22. The ‘‘moreover’’ statement is clear.  $\square$

Let  $R$  be a 1-PVD (i.e., PVD) and  $P$  a prime ideal of  $R$ . Then  $A_1(P) = P$ ; so  $V = (A_1(P) : A_1(P)) = (P : P)$  is a 1-VD (i.e., valuation domain) by [8, Proposition 4.3], and it is easily checked that  $P$  is the maximal ideal of  $V$ . We have the following analogous result for  $n$ -PVDs.

**Theorem 4.32.** *Let  $R$  is an  $n$ -PVD,  $P$  a prime ideal of  $R$ , and  $I = (A_n(P))$ . Then  $V = (I : I)$  is an  $n$ -VD with maximal ideal  $\sqrt{IV} = \{x \in V \mid x^n \in I\}$ . Moreover,  $\sqrt{IV} = \{x \in V \mid x^n \in P\} = \sqrt{PV}$ .*

*Proof.* Let  $x \in K$  with  $x^n \notin V$ . Then  $x^n \notin P$ ; so  $x^{-n}I \subseteq I$  by Corollary 4.15. Thus  $x^{-n} \in V$ ; so  $V$  is an  $n$ -VD with maximal ideal  $N_V$ . Let  $y \in N_V$ . Assume that  $y^n \notin I$ ; so  $y^n \notin P$ . Thus  $y^{-n}I \subseteq I$  by Corollary 4.15 again; so  $y^{-n} \in V$ . Hence  $y \in U(V)$ , a contradiction. Thus  $N_V \subseteq \{x \in V \mid x^n \in I\} \subseteq \sqrt{IV}$ . Also,  $IV = I \subseteq V$ ; so  $\sqrt{IV} \subseteq N_V$ . Hence  $N_V = \sqrt{IV} = \{x \in V \mid x^n \in I\}$ . Clearly  $\{x \in V \mid x^n \in I\} \subseteq \{x \in V \mid x^n \in P\}$  since  $I \subseteq P$ . Also,  $x^n \in P$  for  $x \in V \Rightarrow x^n \in A_n(P)$ ; so  $\{x \in V \mid x^n \in P\} \subseteq \{x \in V \mid x^n \in I\}$ . Thus  $\{x \in V \mid x^n \in I\} = \{x \in$

$V \mid x^n \in P\}$ . Clearly  $x \in P \Rightarrow x^n \in A_n(P) \subseteq I \Rightarrow x^n \in \sqrt{IV}$ ; so  $P \subseteq \sqrt{IV}$ , and hence  $\sqrt{PV} \subseteq \sqrt{IV}$ . Also,  $\sqrt{IV} \subseteq \sqrt{PV}$  since  $I \subseteq P$ ; so  $\sqrt{IV} = \sqrt{PV}$ .  $\square$

Recall that a quasilocal integral domain  $R$  with maximal ideal  $M$  is a PVD if and only if  $(M : M)$  is a valuation domain with maximal ideal  $M$  [11, Proposition 2.5]. Example 4.34(c) below shows that if  $R$  is an  $n$ -PVD with maximal ideal  $M$ , then  $(M : M)$  need not be an  $n$ -VD. And Example 4.34(d)(e) shows that  $V = (M : M)$  may be an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$  when  $R$  is not an  $n$ -PVD. However, since  $M = A_1(M)$ , the next theorem may be viewed as the  $n$ -PVD analog. By adding the extra condition “ $(*)_n$ : if  $x \in K$  is a nonunit of  $\bar{R}$ , then  $x^n \in M$ ,” we get a converse to Theorem 4.32. Note that  $I = (A_n(M)) \subsetneq M$  in general (see Example 4.34(a)(b)).

**Theorem 4.33.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$ , quotient field  $K$ , and  $I = (A_n(M))$ . Then the following statements are equivalent.*

- (1)  $R$  is an  $n$ -PVD.
- (2)  $V = (I : I)$  is an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ , and if  $x \in K$  is a nonunit of  $\bar{R}$ , then  $x^n \in M$ .

*Proof.* (1)  $\Rightarrow$  (2) By Theorem 4.32,  $V$  is an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ . Let  $x \in K$  be a nonunit of  $\bar{R}$ . Then  $x^n \in M$  by Corollary 4.27.

(2)  $\Rightarrow$  (1) Let  $x \in K$ . Suppose that  $x \in E_n(M)$ , i.e.,  $x^n \notin M$ . First, assume that  $x^n \in V$ . Suppose that  $x^n \in N = \{x \in V \mid x^n \in M\}$ ; so  $x^{n^2} = (x^n)^n \in M$ . Thus  $x \in \bar{R}$  and  $x$  is a nonunit of  $\bar{R}$ ; so  $x^n \in M$  by hypothesis, a contradiction. Hence  $x^n \in U(V)$ , and thus  $x^{-n}I \subseteq I$ . Hence  $x^{-n}d \in I \subseteq M$  for every  $d \in A_n(M)$ . Now, suppose that  $x^n \notin V$ . Then  $x^{-n} \in V$  since  $V$  is an  $n$ -VD. Thus  $x^{-n}I \subseteq I$ , and hence  $x^{-n}d \in I \subseteq M$  for every  $d \in A_n(M)$ . Thus  $x^{-n}d \in M$  for every  $x \in E_n(M)$  and  $d \in A_n(M)$ ; so  $R$  is an  $n$ -PVD by Corollary 4.16.  $\square$

We end this section with several examples.

**Example 4.34.** (a) Let  $R = \mathbb{Z}_2[[X^2, X^3]] = \mathbb{Z}_2 + X^2\mathbb{Z}_2[[X]]$ . Then  $R$  is quasilocal with maximal ideal  $M = (X^2, X^3) = X^2\mathbb{Z}_2[[X]]$  and quotient field  $K = \mathbb{Z}_2[[X]][1/X]$ . It is easily checked that  $R$  is an  $n$ -PVD if and only if  $n \geq 2$  and an  $n$ -VD if and only if  $n$  is even. First, suppose that  $n$  is even. Then  $I = (A_n(M)) = \mathbb{Z}_2X^n + X^{n+2}\mathbb{Z}_2[[X]] \subsetneq M$  and  $V = (I : I) = R$  has maximal ideal  $M_V = M$ . Also,  $M_V = \{x \in V \mid x^n \in M\} \subsetneq \{x \in K \mid x^n \in M\} = X\mathbb{Z}_2[[X]]$ . Next, suppose that  $n \geq 3$  is odd. Then  $I = (A_n(M)) = X^n\mathbb{Z}_2[[X]] \subsetneq M$  and  $V = (I : I) = \mathbb{Z}_2[[X]]$  has maximal ideal  $M_V = X\mathbb{Z}_2[[X]] = \{x \in K \mid x^n \in M\}$ .

(b) Let  $R = F[[X^2, X^3]] = F + X^2F[[X]]$ , where  $F$  is a field. Then  $R$  is quasilocal with maximal ideal  $M = (X^2, X^3) = X^2F[[X]]$  and quotient field  $F[[X]][1/X]$ , and  $R$  is an  $n$ -PVD if and only if  $n \geq 2$ . If  $\text{char}(F) = 2$ , then  $(A_n(M)) \subsetneq M$  for every integer  $n \geq 2$ . However,  $M = (A_2(M))$  if  $\text{char}(F) \neq 2$ .

(c) Let  $R = \mathbb{Z}_p + \mathbb{Z}_pX + X^2F[[X]]$ , where  $F = \bar{\mathbb{Z}}_p$  is the algebraic closure of  $\mathbb{Z}_p$ . Then  $R$  is quasilocal with maximal ideal  $M = \mathbb{Z}_pX + X^2F[[X]]$  and quotient field  $K = F[[X]][1/X]$ . Moreover,  $R$  is an  $n$ -PVD if and only if  $n \geq 2$  by Theorem 4.29 since  $\bar{R} = F[[X]]$  is a PVD (in fact, a valuation domain). However,  $V = (M : M) = \mathbb{Z}_p + XF[[X]]$  is an almost valuation domain with maximal ideal  $XF[[X]] = \{x \in K \mid x^n \in M\}$ , but  $V$  is not an  $n$ -VD for any positive integer  $n$  by Example 4.21(b). Note that  $V$  is a PVD, and thus an  $n$ -PVD for every positive integer  $n$ .

(d) Let  $F$  be a field and  $N$  a positive integer. Then  $R_N = F + X^N F[[X]]$  is a quasilocal integral domain with maximal ideal  $M_N = X^N F[[X]]$ , quotient field  $F[[X]][1/X]$ , and integral closure  $\overline{R_N} = F[[X]]$ . Note that  $V_N = (M_N : M_N) = F[[X]]$  is a valuation domain with maximal ideal  $X F[[X]] = \{x \in V_N \mid x^N \in M_N\} = \sqrt{M_N V_N}$ , and thus  $V_N$  is an  $n$ -VD for every positive integer  $n$ . However,  $R_N$  is an  $n$ -PVD if and only if  $n \geq N$ , and  $R_N$  satisfies condition  $(*)_n$  if and only if  $n \geq N$ .

(e) Let  $R = \mathbb{Z}_3 + \mathbb{Z}_3 X^9 + X^{12} \mathbb{Z}_3[[X]]$ . Then  $R$  is a quasilocal integral domain with maximal ideal  $M = \mathbb{Z}_3 X^9 + X^{12} \mathbb{Z}_3[[X]]$ , quotient field  $\mathbb{Z}_3[[X]][1/X]$ , and integral closure  $\overline{R} = \mathbb{Z}_3[[X]]$ . Note that  $V = (M : M) = \mathbb{Z}_3 + X^3 \mathbb{Z}_3[[X]]$  is a 3-VD with maximal ideal  $X^3 \mathbb{Z}_3[[X]] = \sqrt{M V} = \{x \in V \mid x^3 \in M\}$ . However,  $R$  is not a 3-PVD since  $(X^2)^3 (X^2)^3 \in M$ , but  $X^6 \notin M$ , and  $R$  does not satisfy condition  $(*)_3$  since  $X^3 \notin M$ .

## 5. PSEUDO $n$ -STRONGLY PRIME IDEALS, P $n$ VDS, AND $n$ -VDS

In this final section, we introduce and investigate pseudo  $n$ -valuation domains (P $n$ VDS), yet another generalization of PVDs. We also give some more results on  $n$ -VDS.

Let  $R$  be an integral domain with quotient field  $K$ . Recall [16] that  $R$  is a *pseudo-almost valuation domain* (PAVD) if every prime ideal  $P$  of  $R$  is *pseudo-strongly prime*, i.e., if whenever  $xyP \subseteq P$  for  $x, y \in K$ , then there is a positive integer  $n$  such that  $x^n \in R$  or  $y^n P \subseteq P$ . Also, recall [17] that  $R$  is an *almost pseudo-valuation domain* (APVD) if every prime ideal  $P$  of  $R$  is *strongly primary*, i.e., if whenever  $xy \in P$  for  $x, y \in K$ , then  $x^n \in P$  for some positive integer  $n$  or  $y \in P$ . Note that valuation domain  $\Rightarrow$  PVD  $\Rightarrow$  APVD  $\Rightarrow$  PAVD, and no implication is reversible [16, page 1168].

The following is an example of an  $n$ -PVD for some integer  $n \geq 2$  which is neither an APVD, a PAVD, a PVD, nor an almost valuation domain.

**Example 5.1.** (cf. [16, Example 3.4]) Let  $R = \mathbb{Q} + \mathbb{C}X^2 + X^4 \mathbb{C}[[X]]$ . Then  $R$  is quasilocal with maximal ideal  $M = \mathbb{C}X^2 + X^4 \mathbb{C}[[X]]$  and quotient field  $K = \mathbb{C}[[X]][1/X]$ . One can see that  $R$  is neither an APVD, a PAVD, a PVD, an almost valuation domain, nor an  $n$ -VD for any positive integer  $n$ . However, it is easily checked that  $R$  is a  $n$ -PVD for  $n \geq 4$  and  $\overline{R} = \overline{\mathbb{Q}} + X \mathbb{C}[[X]]$  is a PVD with maximal ideal  $N = \{x \in K \mid x^4 \in M\} = X \mathbb{C}[[X]]$ , where  $\overline{\mathbb{Q}}$  is the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . Note that  $\overline{R}$  is not a valuation domain; in fact,  $\overline{R}$  is not an  $n$ -VD for any positive integer  $n$ , and  $R$  is not an  $n$ -PVD for  $n = 1, 2$ , or  $3$ .

We now give yet another “ $n$ ” generalization of PVDs.

**Definition 5.2.** Let  $R$  an integral domain with quotient field  $K$ . A prime ideal  $P$  of  $R$  is a *pseudo  $n$ -strongly prime ideal* of  $R$  if whenever  $xyP \subseteq P$  for  $x, y \in K$ , then  $x^n \in R$  or  $y^n P \subseteq P$ . If every prime ideal of  $R$  is a pseudo  $n$ -strongly prime ideal of  $R$ , then  $R$  is a *pseudo  $n$ -valuation domain* (P $n$ VD).

A P1VD is just a PVD [24, Proposition 1.2], an  $n$ -VD is a P $n$ VD, a P $n$ VD is a PAVD, and a P $n$ VD is also a P( $mn$ )VD for every positive integer  $m$ . Moreover, from Theorem 5.4 and Remark 5.6, it follows that a P $n$ VD  $R$  is quasilocal, the prime ideals of  $R$  are linearly ordered by inclusion, and  $\dim(R) \leq 1$  when  $R$  is Noetherian.

The following is an example of a PAVD which is not a  $Pn$ VD for any positive integer  $n$ .

**Example 5.3.** Let  $p$  be a positive prime integer and  $F = \overline{\mathbb{Z}_p}$  the algebraic closure of  $\mathbb{Z}_p$ . Then  $R = \mathbb{Z}_p + \mathbb{Z}_p X + X^2 F[[X]]$  is quasilocal with maximal ideal  $M = \mathbb{Z}_p X + X^2 F[[X]]$  and quotient field  $K = F[[X]][1/X]$ . Let  $y \in K$  with  $y^n \notin R$  for every positive integer  $n$ . Then  $y = z/X^m$ , where  $z \in U(F[[X]])$  and  $m \geq 0$ . If  $m > 0$ , then  $y^{-2}M \subseteq M$ . If  $m = 0$ , then there is a positive integer  $n$  such that  $z(0)^n = 1$ ; so  $y^{-n}M \subseteq M$ . Thus  $R$  is a PAVD by [16, Lemma 2.1 and Theorem 2.5]. We now show that  $R$  is not a  $Pn$ VD for any positive integer  $n$ . For  $n$  a positive integer, there is a  $b \in F$  with  $b^n \notin \mathbb{Z}_p$  and  $b^{-n} \notin \mathbb{Z}_p$ . Hence  $b^n \notin R$  and  $b^{-n}X \notin M$ ; so  $R$  is not a  $Pn$ VD by Theorem 5.4(a)(b) below. However,  $R$  is an  $n$ -PVD for every integer  $n \geq 2$  by Example 4.34(c).

The proofs of the following results are similar to the proofs given in [16], and thus the details are left to the reader.

**Theorem 5.4.** *Let  $R$  an integral domain with quotient field  $K$ .*

(a) *Let  $P$  be a prime ideal of  $R$ . Then  $P$  is a pseudo  $n$ -strongly prime ideal of  $R$  if and only if  $x^{-n}P \subseteq P$  for every  $x \in E_n(R)$  (see [16, Lemma 2.1]).*

(b)  *$R$  is a  $Pn$ VD if and only if  $R$  is quasilocal with pseudo  $n$ -strongly prime maximal ideal (see [16, Theorem 2.5]).*

(c)  *$R$  is a  $Pn$ VD if and only if for every  $a, b \in R$ , we have  $a^n \mid b^n$  in  $R$  or  $b^n \mid a^n c$  in  $R$  for every nonunit  $c$  of  $R$  (see [16, Proposition 2.9]).*

(d) *Let  $P$  be a prime ideal of  $R$ . If  $R$  is a  $Pn$ VD, then  $R/P$  is a  $Pn$ VD (see [16, Proposition 2.14]).*

(d) *An  $n$ -root closed  $Pn$ VD is a PVD (see [16, Theorem 2.13]).*

The next example gives some more  $n$ -PVDs that are not  $Pn$ VDs.

**Example 5.5.** Let  $m \geq 2$  be an integer. Then  $R = \mathbb{R} + \mathbb{R}X^{m-1} + X^m \mathbb{C}[[X]]$  is quasilocal with maximal ideal  $M = \mathbb{R}X^{m-1} + X^m \mathbb{C}[[X]]$ , quotient field  $K = \mathbb{C}[[X]][1/X]$ , and integral closure  $\bar{R} = \mathbb{C}[[X]]$ . By Theorem 4.29,  $R$  is an  $n$ -PVD for every integer  $n \geq m$ . For a positive integer  $k$ , let  $y = e^{-i\pi/2k}$ . Then  $y^k = -i \notin R$  and  $y^{-k}X^{m-1} = iX^{m-1} \notin R$ ; so  $R$  is not a  $Pk$ VD for any positive integer  $k$  by Theorem 5.4(a).

**Remark 5.6.** Let  $R$  an integral domain with quotient field  $K$ . Since  $A_n(P) \subseteq P$  for every prime ideal  $P$  of  $R$ , every pseudo  $n$ -strongly prime ideal of  $R$  is also an  $n$ -powerful semiprimary ideal of  $R$  by Corollary 4.15 and Theorem 5.4(a), and thus a  $Pn$ VD is an  $n$ -PVD. Hence, we have the following implications

$$n\text{-VD} \Rightarrow Pn\text{VD} \Rightarrow n\text{-PVD}.$$

Neither of the above two implications is reversible. A  $Pn$ VD need not be an  $n$ -VD by Theorem 5.13, and an  $n$ -PVD need not be a  $Pn$ VD by Examples 5.3 and 5.5. Also, note that the ring in Example 5.1 is a 4-PVD, but not a P4VD.

The next theorem gives a case where an  $n$ -PVD is a  $Pn$ VD. Note that the  $n = 1$  case is just [11, Proposition 2.5]. We may have  $M \neq (A_n(M))$  for every integer  $n \geq 2$  (see Example 4.34(a)(b)). Note that in the next two theorems, we need the extra condition  $(*)_n$  (cf. Example 4.34(d)(e)), and recall that if  $R$  is not an  $n$ -PVD, then  $R$  is not a  $Pn$ VD by Remark 5.6).

**Theorem 5.7.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M = (A_n(M))$  and quotient field  $K$ . Then the following statements are equivalent.*

- (1)  $R$  is a PnVD.
- (2)  $R$  is an  $n$ -PVD.
- (3)  $V = (M : M)$  is an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ , and if  $x \in K$  is a nonunit of  $\bar{R}$ , then  $x^n \in M$ .

*Proof.* (1)  $\Rightarrow$  (2) A PnVD is an  $n$ -PVD by Remark 5.6.

(2)  $\Rightarrow$  (1) Let  $x \in E_n(R)$ ; so  $x \in E_n(M)$ . Then  $x^{-n}A_n(M) \subseteq M$  by Corollary 4.16, and thus  $x^{-n}M \subseteq M$  since  $M = (A_n(M))$  by hypothesis. Hence  $R$  is a PnVD by Theorem 5.4(a)(b).

(2)  $\Leftrightarrow$  (3) This is clear by Theorem 4.33.  $\square$

The following result recovers that a quasilocal integral domain  $R$  with maximal ideal  $M$  is a PVD if and only if  $(M : M)$  is a valuation domain with maximal ideal  $M$  [11, Proposition 2.5]; its proof is an analog of the proof of [16, Theorem 2.15].

**Theorem 5.8.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ . Then the following statements are equivalent.*

- (1)  $R$  is a PnVD.
- (2)  $V = (M : M)$  is an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ , and if  $x \in K$  is a nonunit of  $\bar{R}$ , then  $x^n \in M$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $R$  be a PnVD. Let  $x \in E_n(V)$ ; so  $x \in E_n(R)$ . Then  $x^{-n}M \subseteq M$  by Theorem 5.4(a); so  $x^{-n} \in V$ . Thus  $V$  is an  $n$ -VD with maximal ideal  $M_V$ . Let  $x \in M_V$ . If  $x^n \in R$ , then  $x^n \in M$ . Otherwise,  $x \in E_n(R)$ . Hence  $x^{-n}M \subseteq M$  by Theorem 5.4(a) again; so  $x^{-n} \in V$ . Thus  $x \in U(V)$ , a contradiction. Hence  $M_V \subseteq \{x \in V \mid x^n \in M\} \subseteq \sqrt{MV}$ , and  $\sqrt{MV} \subseteq M_V$  since  $MV = M \subsetneq V$ . Thus  $M_V = \sqrt{MV} = \{x \in V \mid x^n \in M\}$ . If  $x \in K$  is a nonunit of  $\bar{R}$ , then  $x^n \in M$  by Corollary 4.27 since a PnVD is an  $n$ -PVD by Remark 5.6.

(2)  $\Rightarrow$  (1) Let  $V = (M : M)$  be an  $n$ -VD with maximal ideal  $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ . Suppose that  $x \in E_n(R)$ ; so  $x^n \notin M$ . If  $x^n \in V$  and  $x^n \notin U(V)$ , then  $x^{n^2} = (x^n)^n \in M \subseteq R$ ; so  $x \in \bar{R}$ . Thus  $x^n \in M$  by hypothesis, a contradiction. Hence  $x^n \in U(V)$ ; so  $x^{-n}M \subseteq M$ . If  $x^n \notin V$ , then  $x^{-n} \in V$  since  $V$  is an  $n$ -VD. Thus  $x^{-n}M \subseteq M$  in either case; so  $R$  is a PnVD by Theorem 5.4(a)(b).  $\square$

**Corollary 5.9.** *Let  $R$  be a PnVD with maximal ideal  $M$ . If  $P$  is a prime ideal of  $R$ , then  $W_P = (P : P)$  is an  $n$ -VD. Moreover, if  $P \subseteq Q$  are prime ideals of  $R$ , then  $W_Q = (Q : Q) \subseteq (P : P) = W_P$ .*

*Proof.* We have  $V = (M : M) \subseteq (P : P) = W_P$  by [6, Lemma 2.2] since  $P \subseteq M$ . Thus  $W_P$  is an  $n$ -VD since  $V$  is an  $n$ -VD by Theorem 5.8. The ‘‘moreover’’ statement is clear since  $(Q : Q) \subseteq (P : P)$  by [6, Lemma 2.2] again.  $\square$

Let  $T$  be an overring of an integral domain  $R$  and  $n$  a positive integer. Then  $T$  is an  $n$ -root extension of  $R$  if  $x^n \in R$  for every  $x \in T$ , and  $T$  is a root extension of  $R$  if for every  $x \in T$ , there is a positive integer  $m$  such that  $x^m \in R$ .

**Theorem 5.10.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ ,  $n$  a positive integer, and  $V$  a valuation overring of  $R$  with maximal ideal  $N = \{x \in V \mid x^n \in M\}$ . Then  $R$  is an  $n$ -VD if and only if  $V$  is an  $n$ -root extension of  $R$ .*

*Proof.* We may assume that  $R \subsetneq V$ . Suppose that  $R$  is an  $n$ -VD. Let  $x \in V \setminus R$ . If  $x \in N$ , then  $x^n \in M \subseteq R$ . Thus, assume that  $x \notin N$ . Since  $N$  is the maximal ideal of  $V$ , we have  $x \in U(V)$ . Thus  $x^n \notin M$  and  $x^{-n} \notin M$ . Since  $R$  is an  $n$ -VD, we have  $x^n \in U(R) \subseteq R$ . Hence  $V$  is an  $n$ -root extension of  $R$ .

Conversely, suppose that  $V$  is an  $n$ -root extension of  $R$ . Let  $x \in K$  with  $x^n \notin R$ . Then  $x \notin V$  since  $V$  is an  $n$ -root extension of  $R$ , and thus  $x^{-1} \in V$  since  $V$  is a valuation domain. Hence  $x^{-n} \in R$  since  $V$  is an  $n$ -root extension of  $R$ , and thus  $R$  is an  $n$ -VD.  $\square$

**Lemma 5.11.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ . If  $R$  is an  $n$ -VD, then  $\overline{R}$  is a valuation domain with maximal ideal  $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$  and  $R \subseteq \overline{R}$  is an  $n$ -root extension.*

*Proof.* Let  $R$  be an  $n$ -VD. Then  $R$  is an almost valuation domain; so  $\overline{R}$  is a valuation domain and  $R \subseteq \overline{R}$  is a root extension by [4, Theorem 5.6]. Thus  $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$  is the maximal ideal of  $\overline{R}$  by Theorem 4.29 since an  $n$ -VD is an  $n$ -PVD. Hence  $\overline{R}$  is an  $n$ -root extension of  $R$  by Theorem 5.10.  $\square$

**Theorem 5.12.** *Let  $R$  be a quasilocal integral domain with maximal ideal  $M$  and quotient field  $K$ , and let  $V$  be an  $n$ -VD overring of  $R$  with maximal ideal  $N = \{x \in V \mid x^n \in M\}$ . Then  $R$  is an  $n$ -VD if and only if  $\overline{V} = \overline{R} = \{x \in K \mid x^n \in R\}$ .*

*Proof.* We may assume that  $R \subsetneq V$ . Suppose that  $R$  is an  $n$ -VD. Then  $\overline{R}$  is a valuation domain with maximal ideal  $W = \{x \in K \mid x^n \in M\}$  and  $R \subseteq \overline{R}$  is an  $n$ -root extension by Lemma 5.11. Similarly, since  $V$  is an  $n$ -VD,  $\overline{V}$  is a valuation domain with maximal ideal  $T = \{x \in K \mid x^n \in N\}$  and  $V \subseteq \overline{V}$  is an  $n$ -root extension by Lemma 5.11. First, we show that  $R \subsetneq V$  is an  $n$ -root extension. Let  $x \in V \setminus R$ . If  $x \in N$ , then  $x^n \in M \subseteq R$ . Hence, assume that  $x \notin N$ . Since  $N$  is the maximal ideal of  $V$ , we have  $x \in U(V)$ . Since  $x \in U(V)$ , neither  $x^n \in M$  nor  $x^{-n} \in M$ . Since  $R$  is an  $n$ -VD,  $x^n \in U(R) \subseteq R$ . Thus  $V$  is an  $n$ -root extension of  $R$ . Since  $V$  is an integral overring of  $R$ , we have that  $\overline{V}$  is integral over  $R$ , and thus  $\overline{R} = \overline{V} = \{x \in K \mid x^n \in R\}$ .

Conversely, suppose that  $\overline{R} = \overline{V} = \{x \in K \mid x^n \in R\}$ , and let  $x \in K$  with  $x^n \notin R$ . Then  $x \notin \overline{V}$ , and thus  $x^{-1} \in \overline{V}$  since  $\overline{V}$  is a valuation domain by Lemma 5.11. Hence  $x^{-n} \in R$ ; so  $R$  is an  $n$ -VD.  $\square$

Let  $V$  be a valuation domain with maximal ideal  $M$ , residue field  $F = V/M$ , and  $\pi : V \rightarrow F$  the canonical epimorphism. If  $k$  is a subfield of  $F$ , then  $R = \pi^{-1}(k)$  is a PVD with maximal ideal  $M$  [11, Proposition 2.6]. Moreover, every PVD arises in this way. Let  $R$  be a PVD with maximal ideal  $M$ . Then  $V = (M : M)$  is a valuation domain with maximal ideal  $M$  [11, Proposition 2.5]; so  $R = \pi^{-1}(R/M)$ . A similar result holds for PnVDs and  $n$ -VDs.

**Theorem 5.13.** *Let  $V$  be an  $n$ -VD with nonzero maximal ideal  $M$ , residue field  $F = V/M$ ,  $\pi : V \rightarrow F$  the canonical epimorphism,  $k$  a subfield of  $F$ , and  $R = \pi^{-1}(k)$ . Then the pullback  $R = V \times_F k$  is a PnVD with maximal ideal  $M$ . In particular, if  $k$  is properly contained in  $F$  and  $V$  is not an  $n$ -root extension of  $R$ , then  $R$  is a PnVD which is not an  $n$ -VD.*

*Proof.* In view of the construction stated in the hypothesis, it is well known that  $M$  is a maximal ideal of  $R$  for any integral domain  $V$ . Also, it is clear that  $R$  and  $V$  have the same quotient field  $K$  by [11, Lemma 3.1]. Let  $x \in E_n(R)$ . Then

$x^n \in V$  or  $x^{-n} \in V$  since  $V$  is an  $n$ -VD. Suppose that  $x^n \in V$ . Since  $x \in E_n(R)$  and  $M$  is the maximal ideal of  $R$ , we have  $x^n \notin M$ . Thus  $x^n \in U(V)$ , and hence  $x^{-n} \in V$ ; so  $x^{-n}M \subseteq M$  since  $M$  is an ideal of  $V$ . Now suppose that  $x^{-n} \in V$ . Then  $x^{-n}M \subseteq M$  since  $M$  is an ideal of  $V$ . Thus  $M$  is a pseudo  $n$ -strongly prime ideal of  $R$  by Theorem 5.4(a), and hence  $R$  is a PnVD by Theorem 5.4(b). The remaining part is clear from Theorem 5.12.  $\square$

The final example illustrates the previous theorem.

**Example 5.14.** (a) Let  $V = \mathbb{Z}_p(t)[[X]]$ . Then  $V$  is a valuation domain; so  $R = \mathbb{Z}_p + X\mathbb{Z}_p(t)[[X]]$  is a PnVD for every positive integer  $n$ , but not an  $n$ -VD for any positive integer  $n$ , by Theorem 5.13 since  $V$  is not an  $n$ -root extension of  $R$ . Note that  $R$  is actually a PVD.

(b) Let  $T = K + M$  be a quasilocal integral domain with maximal ideal  $M$  and  $K$  a subfield of  $T$ . Let  $k$  be a subfield of  $K$  and  $R = k + M$ . Then  $R$  is also quasilocal with maximal ideal  $M$ . Thus  $R$  is an  $n$ -PVD (resp., PnVD) if and only if  $T$  is an  $n$ -PVD (resp., PnVD) by Corollary 4.8 (resp., Theorem 5.4(b)).

For example,  $T = \mathbb{R}[[X^2, X^3]] = \mathbb{R} + X^2\mathbb{R}[[X]]$  is an  $n$ -PVD  $\Leftrightarrow n \geq 2$  (Example 4.34(b)), and thus  $R = \mathbb{Q} + X^2\mathbb{R}[[X]]$  is an  $n$ -PVD  $\Leftrightarrow n \geq 2$ .

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## REFERENCES

- [1] D. D. Anderson and D. F. Anderson, Multiplicatively closed subsets of fields, *Houston J. Math.* 13(1987), 1–11.
- [2] D. D. Anderson and M. Winders, Idealization of a module, *J. Commut. Algebra* 1(2009), 3–56.
- [3] D. D. Anderson, K. R. Knopp. and R. L. Lewin, Almost Bezout domains, II, *J. Algebra* 167(1994), 547–556.
- [4] D. D. Anderson and M. Zafrullah, Almost Bezout domains, *J. Algebra* 142(1991), 285–309.
- [5] D. D. Anderson and M. Zafrullah, Almost Bezout domains, III, *Bull. Math. Soc. Sci. Math. Roumanie* 51(2008), 3–9.
- [6] D. F. Anderson, Comparability of ideals and valuation overrings, *Houston J. Math.* 5(1979), 451–463.
- [7] D. F. Anderson, Root closure in integral domains, *J. Algebra* 79(1982), 51–59.
- [8] D. F. Anderson, When the dual of an ideal is a ring, *Houston J. Math.* 9(1983), 325–332.
- [9] D. F. Anderson and A. Badawi, On  $n$ -absorbing ideals of commutative rings, *Comm. Algebra* 39(2011), 1646–1672.
- [10] D. F. Anderson and A. Badawi, On  $(m,n)$ -closed ideals of commutative rings, *J. Algebra Appl.* 16(2017)(21 pages).
- [11] D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, *Canad. J. Math.* 32(1980), 362–384.
- [12] D. F. Anderson and D. E. Dobbs, On the product of ideals, *Canad. Math. Bull.* 26(1983), 238–246.
- [13] A. Badawi, On abelian  $\pi$ -regular rings, *Comm. Algebra* 25(1997), 1009–1021.
- [14] A. Badawi, On divided commutative rings, *Comm. Algebra* 27(1999), 1465–1474.
- [15] A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.* 75 (2007), 417–429.
- [16] A. Badawi, On pseudo-almost valuation domains, *Comm. Algebra* 35(2007), 1167–1181.
- [17] A. Badawi and E. Houston, Powerful ideals, strongly primary ideals, pseudo-valuation domains, and conducive domains, *Comm. Algebra* 30(2002), 1591–1606.
- [18] H. S. Choi and A. Walker, The radical of an  $n$ -absorbing ideal, arXiv:1610.10077 [math.AC], *J. Commut. Algebra*, to appear.

- [19] J. A. Cox and A. J. Hetzel, Uniformly primary ideals, *J. Pure. Appl. Algebra* 212(2008), 1–8.
- [20] D. E. Dobbs, Divided rings and going-down, *Pacific J. Math* 67(1976), 353–363.
- [21] D. E. Dobbs, Coherence, ascent of going-down, and pseudo-valuation domains, *Houston J. Math.* 4(1978), 551–576.
- [22] G. Donadze, A proof of the Anderson–Badawi  $\text{rad}(I)^n \subseteq I$  formula for  $n$ -absorbing ideals, *Proc. Indian Acad. Sci. (Math. Sci.)* 128(2018), 6–11.
- [23] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.
- [24] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, *Pacific J. Math.* 75(1978), 137–147.
- [25] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II, *Houston J. Math.* 4(1978), 199–207.
- [26] J. Huckaba, *Rings with Zero Divisors*, Marcel Dekker, Inc., New York, 1988.
- [27] I. Kaplansky, *Commutative Rings, Revised Edition*, The University of Chicago Press, Chicago and London, 1974.
- [28] M. D. Larson and P. J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York/London, 1971.
- [29] N. Mahdou, A. Mimouni, and M. Moutui, On almost valuation and almost Bezout rings, *Comm. Algebra* 43(2015), 297–308.

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