Three-Stage Forward and Closed Loop Supply Chain Models Under Consignment Stock Partnership

THREE-STAGE FORWARD AND CLOSED LOOP SUPPLY CHAIN MODELS UNDER CONSIGNMENT STOCK PARTNERSHIP

by

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Approval Signatures

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Dedication

This thesis is dedicated to my lovely father, Mr. Samih Khader Alkhatib, my mother and all my brothers and sisters. Thank you all for being the source of trust and support whenever needed. I highly appreciate your encouragement and unconditional love throughout my thesis work and personal life.

Abstract

Achieving operational excellence is a vital goal ought to be emphasized throughout supply chain (SC) practices nowadays. The integration of raw material procurement decisions at the upstream stages, along with the production and dispatching decisions at the downstream stages, greatly impacts the production schedule and chain-wide total cost. The significance of this research work is that it is the first to address and propose mathematical models for a three-stage forward and closed loop supply chain systems operating under consignment stock (CS) partnership in which the inbound and outbound logistical decisions are simultaneously accounted for, coupled with the production sequencing decisions at the vendor stage. The objective is to optimize the raw material replenishment schedule (i.e., quantity and frequency) as well as the production sequence and the delivery schedule of the finished product to the downstream buyer. Throughout this work, mixed integer non-linear programming (MINLP) models are developed that jointly seek to optimize the procurement decisions of raw material, the length of the production cycle, the sequence to follow in the production of newly and remanufactured batches, number of newly and remanufactured batches produced within one production cycle, as well as the initial inventory levels of recovered and finished products at the vendor's and buyer's premises, respectively, in order to minimize the chain-wide total cost. Extensive numerical experiments are also conducted with the purpose of assessing the impact of key problem parameters on the behavior of the developed models. The three stage forward model generated production schedule with larger batch size when compared to the just-in-time raw material ordering policy for the two-stage forward model. Moreover, the closed loop model economically outperformed the "manufacturing only" policy with cost savings ranging between 7% and 33%. Sensitivity analysis results indicates that the intermittent sequences, in which the setup for either newly manufactured or remanufactured products takes place more than once, are economically preferable in the case of high remanufacturing and low manufacturing setup costs.

Search Terms: Three-stage closed loop supply chain, three-stage forward supply chain, consignment stock, raw material ordering policies, Mixed Integer Non-Linear Programming.

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Chapter 1: Introduction

In this chapter, we present an overview of supply chain management, vendor managed inventory, consignment stock, green supply chain, and incorporation of raw material replenishment decisions.

1.1. Supply Chain Overview

Organizations are always seeking to improve their internal processes in order to have a competitive advantage over their competitors. Achieving cost reduction and profit increase are the two important goals ought to be emphasized throughout this journey of internal process improvement. As such, they are looking to improve their individual overall efficiency where they believed that their success and competitiveness arises best upon locally optimizing their operations. Unfortunately, working only within the company's borders, in complete isolation of the suppliers and the customers, does not yield the anticipated success or the desired optimality. This is attributed to the fact that external factors, beyond the control of the company, greatly affect the individual efficiency and success of the organization. Bearing this in mind, organizations have started to think outside the box through better coordinating their production and inventory related decisions with their partners which allows for a better control over the external factors that are influencing their internal processes. In other words, they started to emphasize on having overall system efficiency and being globally optimum rather than achieving individual efficiency and local optimality. These attempts and thoughts of having overall system efficiency and global optimality have given rise to the term supply chain management (SCM).

A supply chain can be formally defined as:

> *"A network of organizations that are involved, through upstream and downstream linkages in the different processes and activities that produce value in the form of products and services in the hand of the ultimate consumer"* [1]

In a manufacturing settings, all the entities involved in value adding activities while transforming the product from its most basic form as raw material until its delivery to the final customer as a finished product are all members (also referred to as players or partners) in the supply chain. Hence, all the activities along a supply chain should be designed according to the desires and needs of the ultimate customers to be served [1].

Supply chain is about considering all the parties involved, directly or indirectly, towards fulfilling a customer's request. It includes not only the manufacturer and supplier, but also the transporters, warehouses, retailers and even customers themselves [2].

As shown in Figure 1, a typical supply chain consists of five stages with the common possibility of having multiple firms at each stage. These stages are linked by the flow of material, information and funds in the upstream and/or downstream direction. One of the stages, or an intermediate firm may be responsible for managing the flow of the entire chain. These stages include [1]:

- Suppliers
- Manufacturers
- Distributers
- Retailers
- Customers

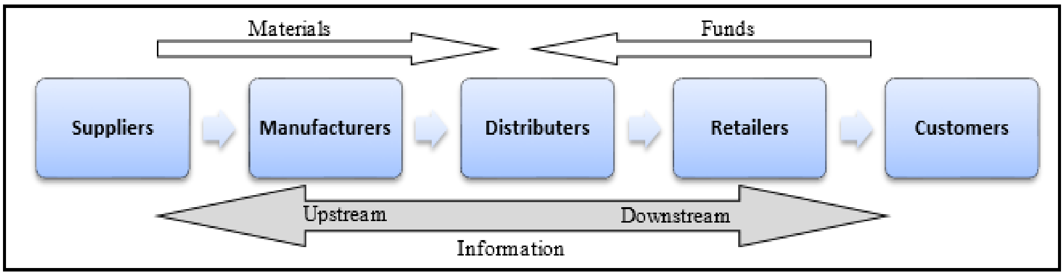

Figure 1: Supply chain stages

As a matter of fact, in any supply chain, customer is the integral part, and satisfying the customer needs is the primary purpose of any firm besides generating profit for itself in the process [1].

Within the broad definition of supply chain, we can define three degrees of supply chain complexity; a direct supply chain, an extended supply chain, and an ultimate supply chain [3]. When the supply chain consists of a company, a supplier and a customer who are involved in the upstream and/or downstream flows, it is a "direct supply chain". However, if suppliers of the immediate supplier and customers of the immediate customers are all involved, it is "an extended supply chain". An ultimate supply chain includes all the organizations involved in all the upstream and downstream flows. Figure 2 shows the three degrees of supply chain complexity [3]:

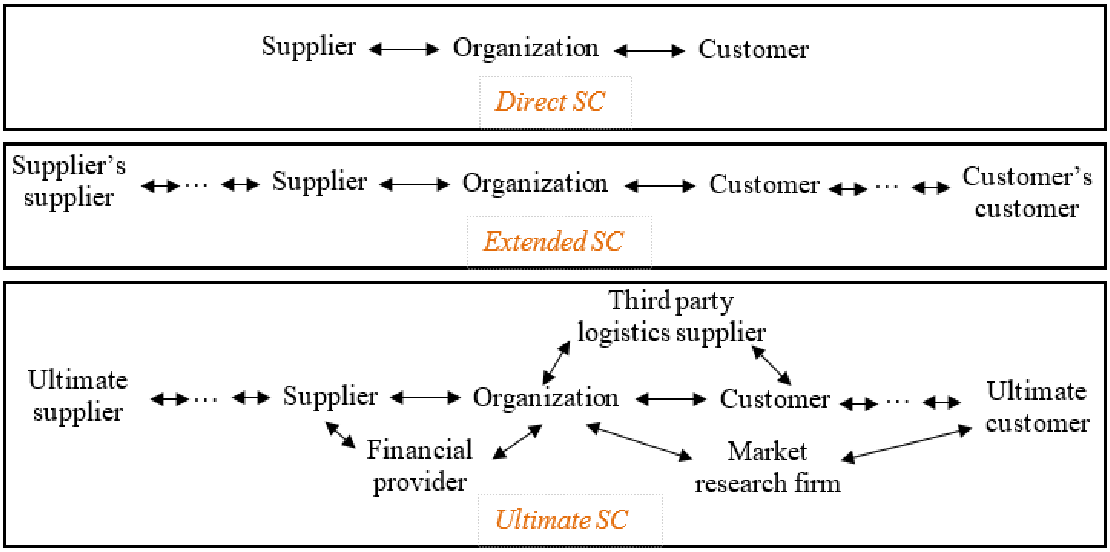

Figure 2: Direct, extended and ultimate supply chain [3]

Another definition of SCM that has been defined by *Stock* [4] that has compiled the most commonly published definitions of SCM is:

> *"The management of a network of relationships within a firm and between interdependent organizations and business units consisting of material suppliers, purchasing, production facilities, logistics, marketing, and related systems that facilitate the forward and reverse flow of materials, services, finances and information from the original producer to final customer with the benefits of adding value, maximizing profitability through efficiencies, and achieving customer satisfaction."*

1.2. Vendor Managed Inventory

As mentioned earlier, being cost efficient is better achieved through the improvement of collaborative planning and integration among supply chain members. In spite the fact that perfectly matching the supply and demand is an impossible process, companies are constantly seeking to keep supply and demand very close. Hence, it is inevitable to hold certain levels of inventories at possibly several locations across the chain. However, firms always try to minimize the costs associated with carrying inventories using supply chain initiatives such as Vendor Managed Inventory (VMI). VMI is a practice that makes the suppliers and vendors of a company responsible for the flow of their goods to the pants of that company, warehouses or retail locations and reduces total supply chain cost [5,6]. Continuous replenishment strategy or VMI agreement is a source of partnership between a buyer and vendor seeking to improve the overall efficiency of the supply chain.

VMI agreement is responsible for reducing two costs associated with inventories; the ordering cost and the holding cost. Ordering cost includes the cost of issuing a purchase order, which might be broken down into order placement cost (initiation cost) and receiving goods cost. On the other hand, the holding cost is the cost associated with maintaining and keeping a stock of goods in storage which includes the capital cost (opportunity cost) and storage related costs [7,8]. VMI agreement shares the benefits of better profitability and efficiency between the two parties, the vendor and the buyer. Indeed, it benefits the buyer in terms of reducing the ordering cost where the initiation cost will be transferred from the buyer to the vendor. In addition, it benefits the vendor by reducing the quantity of items in their inventories and hence reducing the holding cost and transferring it to the buyer. Moreover, VMI benefits the vendor in reducing the uncertainty of demand and in improving the demand’s visibility [9]. Table 1 shows the responsibility of each party for the different aspects of the inventory decision process [11]:

Table 1: Vendor and buyer responsibilities in VMI [11]

Decision		Cost			
Order Quantity	*Number of shipments & timing*	*Ordering cost (shared)*		*Holding cost*	
		Initiation	*Receiving*	*Capital*	*Storage*
Vendor	**Vendor**	**Vendor**	**Buyer**	**Buyer**	**Buyer**

Under VMI agreement, the vendor monitors the buyer’s inventory and becomes responsible for managing it and deciding about the timing and frequency of the shipments and the quantity needed. In other words, the vendor is the one who generates the order and take the responsibility of the initiation cost. However, the ownership of the inventory does not change under VMI and remains under the buyer's ownership that pays for the whole quantity delivered by the vendor at the receiving time. Moreover, sharing of information between the customers and the vendor is required in VMI where the vendor will become able to closely monitor the inventory level of the customer. Using technologies such as Electronic Data Interchange (EDI) or internet will help the vendor in doing demand forecasts and knowing the customer stock levels in advance [10].

1.3. Consignment Stock

Another supply chain initiative is consignment stock (CS) policy, which is similar to VMI in terms of reducing total cost of all parties involved. However, under CS partnership, the buyer is responsible for the decision of shipments timing, frequency and quantity. CS is defined as:

> *"An innovative approach to supply and stock management, based on a strong and continuous collaboration between vendor and buyer to create a 'win-win' situation, where both partners have equal gains"* [12].

Unlike the VMI, the ownership of inventory under CS strategy belongs to the vendor until the items are sold or used by the buyer at which point the vendor gets a share of the revenues. In other words, the buyer will not pay the vendor for the quantity shipped to his inventory stores until he/she makes use of them. Hence, inventory related capital holding costs are charged to the vendor rather than to the buyer. As a result, CS policy reduces the holding cost of both parties by reducing the capital cost, at the buyer's end, and reducing the storage related cost, at the vendor's facility. Moreover, since the decisions regarding timing, frequency and quantity of shipments are handled by the buyer, the vendor is not responsible for the ordering decision and it is rather the buyer who takes care of it. Table 2 shows the responsibility of each party for the different aspects of the inventory decision process [11].

From a vendor's perspective, CS benefits extends to include the optimization of transportation and production lot sizes in addition to having better visibility and less uncertainty about the demand where information of real consumption are available through online data update or EDI. Moreover, more space will be available due to reducing the average stock in the vendor's inventories that can be used for other purposes and may eliminate the necessity of building new warehouses. Another advantage for the vendor due to the use of CS policy is having a long-term relationship, or strategic partnership, with the buyers [12].

Table 2: Vendor and buyer responsibilities in CS partnership [11]

Decision		Cost			
Order Quantity	*Number of shipments & timing*	*Ordering cost*		*Holding cost (shared)*	
		Initiation	*Receiving*	*Capital*	*Storage*
Buyer	**Buyer**	**Buyer**	**Buyer**	**Vendor**	**Buyer**

1.4. Green Supply Chain Management

Protecting the environment and preserving natural resources for future generation is becoming an important concern of people and organizations nowadays. Many regulations released by various governments have forced companies to reduce greenhouse gas emissions and improve waste disposal methods. The drastic increase in the consumption rates of critical resources such as water, food, minerals alongside the increase in the greenhouse emissions have made environmental concerns to become a serious threat. Due to the economic growth that we see nowadays, there is a dramatic increase in industry sectors triggering additional consumption of natural sources and higher levels of greenhouse emissions. In fact, one third of energy consumption in the developed countries is used by industrial firms. Thus, organizations are ought to be more socially and environmentally responsible and try to reduce greenhouse gas emissions. As a matter of fact, there is a great opportunity for them to be environmental friendly [13]. In the U.S only, the total value of products returned by consumers is estimated at $100 billion per annum [14]. Many organizations are now shifting toward green supply chain by integrating the environmental concerns into their supply chain practices.

Globalization has led worldwide organizations to balance their economic and environmental performances in order to achieve a concrete sustainable development [15]. Indeed, integrating green supply chain related practices has increased the competitiveness of several organizations and also led to innovative solutions. Sustainable supply chains or green supply chains are defined as:

> *"Concepts that take a more holistic systems perspective on the total environment impacts of the supply chain on resources and ecological foot prints* [16]*."*

In other words, green supply chain is concerned mainly with the impact on environment in supply chain process and the product design. It includes practices that are environmental friendly such as purchasing green products and selecting and evaluating suppliers that are environmental friendly as well. There are many steps to be taken in order to implement the sustainable supply chain such as environmental sourcing, designing efficient processes and life cycle analysis through energy and material reduction [13]. To avoid the costs of repair, disposal and reuse of products, green product design is being implemented in the early design stage of products [17]. The reverse supply chain is defined as:

> *"A series of activities required to retrieve used products from customers and either dispose of them or reuse them. Reverse supply chain management is defined as the field that studies how to manage those activities effectively and efficiently* [18]"

Reverse supply chain is an effective strategy in achieving social sustainable development and enhancing the market competitiveness of enterprises. It extends the way of operations that traditional supply chain uses, which had an enormous impact on minimizing the overall cost of supply chain as well as increasing the total efficiency in the supply chain and the whole supply chain system [19].

There are considerable economic benefits associated with reverse supply chain. Those benefits provide companies with the enthusiasm and initiative to implement the reverse supply chain. The benefits include renewable raw materials which lower the environmental penalties as it reduces the waste good. In principle, the benefits span over four main areas; the benefits of recycling materials, the benefits of remanufacturing, the benefits of incineration and disposal and the benefits of environmental protection. Some of the benefits that result from the processes of reverse supply chain are reduction of procurement costs of the raw materials, remanufactured product sales revenue, and decreasing the waste emissions [19].

1.5. Integration of Raw Material Procurement Decisions

One of the ultimate strategic goals of most companies is providing a better service with lower cost to the customers. Indeed, high levels of competition among companies as well as the high diversification of products produced with short life cycle have laid the ground for the different companies to look for the different production and inventory related decisions integration. Hence, in order to compete in the market, firms have to operate as an integral part of the supply chain rather than operating individually and autonomously [20].

Across a supply chain, the effective and efficient management of material flows is critical to its success [21]. In fact, in order to produce one product for the end customer, several raw materials are typically required. Hence, the batch quantity of the products determines the ordering quantities of raw materials. In other words, the ordering frequency and quantities of raw materials are dependent on the batch quantity of the product. Hence, the product's optimum production lot size and the associated raw materials' ordering quantities should be determined together. In order to achieve this, production and purchasing could be treated as components of a single system. Normally, the amount of raw materials used to produce a product is known. As such, the amount of raw materials to produce a lot is also known [22].

On one hand, the classical economic order quantity (EOQ) model is used by the firms to determine the optimum ordering quantity (economic ordering lot size) that minimizes the holding cost of the raw materials and the ordering cost of the firm. On the other hand, the classical economic production quantity (EPQ) model is used by the firms to determine the optimum manufacturing quantity (economic manufacturing batch size) that minimizes the holding cost of finished products and the production setup cost of the firm [2]. However, the two models are considered independently when they are used by the firms to deal with their inventory control problems. Nevertheless, previous studies have shown that integrating the inventory control model (inter-firm or intra-firm) will result in a lower total cost for the entire supply chain system than that of planning the models separately [21, 23]. Inter-firm cooperation is integrating firm's production and the ordering of the buyer while intra-firm planning is through integrating firm's raw material procurement and its production [23]

For closed loop supply chains where items are recovered from the market and the serviceable ones are put back into production in what is commonly referred to as remanufacturing, the incorporation of raw material procurement decisions is particularly important. The end customer demand might be fulfilled from (1) newly manufactured products which results from applying value adding activities to the raw material in order to convert them to finished products, or (2) remanufactured products that utilize the returned materials from the market which are reprocessed to transform them to end items. As such, the decision maker at the manufacturing facility is faced with the problem lot sizing related decisions for both newly and remanufactured products as well as the proportion of demand that is to be fulfilled from each of those two product types. Moreover, the important issue in this context is the sequencing of those newly and remanufactured lots. The production sequence determines the level of raw material depletion and accordingly the on-hand inventory as well as the replenishment schedule. Hence, the interrelated planning decisions concerning raw material procurement, production sequencing and lot sizing as well as finished production delivery should be simultaneously accounted for in order to achieve operational excellence and chain wide minimum cost.

1.6. Supply Chain Mathematical Representations

Optimization models are widely used in the supply chain management field, where managing a supply chain, efficiently and effectively, requires dealing with a wide range of input parameters and involves some conflicting tradeoffs that are better decided on through the use of mathematical modeling approach. Some common types of optimization modes are linear programming, mixed-integer programming, stochastic modeling and simulation, as being described below:

- Linear programming (LP): This is one of the most common used optimization models. Resource utilization is one of its many valuable applications that are widely used in many firms that deal with scare resources such as labor, machines, capacity, and materials. LP models optimize the resources in a way that gives the maximum possible profit or the most economical scenario. Personnel scheduling (i.e. labor to shifts assignment) and planning and distribution (i.e. optimizing shipping quantities) problems are typically formulated using LP models.

- Mixed-integer programming (MIP): Such types of models may be viewed as an extension of pure LP models wherein some of the decision variables are defined as integers as they accept only integer values, or in some cases binary variables that assume a value of either zero or one, or a mix of both types of variables, along with some continuous variables. If the objective function and constraints are linear, then such class of models is referred to as mixed integer linear programs (MILPs), whereas in the case of exiting non-linearity, it is referred to as mixed integer non-linear program (MINLPs). Typical examples of MIP models include capacity planning problems, facility location and planning, and transportation modes planning.
- Stochastic programming: These models are commonly used when there is uncertainty in estimating problem parameters such as demand, lead time and cost parameters.

1.7. Problem Statement

The concept of system integration has been widely utilized in the existing literature. The main aim of the relevant research works is to determine optimum integrated policy that leads to minimizing the total chain-wide costs. Many researchers demonstrated the advantages and benefits of implementing an integrated supply chain system rather than operating individually. Some of those researchers studied two-stage supply chain systems while others have contributed in optimizing the system integration of the more involved three-stage supply chain system. Ample amount of the researches has focused on the two-stage system in the forward, reverse, and closed loop logistics. In addition, several researchers have assessed the benefits obtained by all parties upon implementing some of the supply chain initiatives such as CS and VMI.

When addressing three-stage supply chain systems, most of the existing research work has dealt with the problem in the context of forward supply chain systems taking into account the procurement of raw materials in their decisions. However, none of the previous studies have addressed integrating a three-stage forward or closed loop supply chain system with CS partnership. Thus, this research work stands out as the first to present integrated models for a three stage forward as well as closed loop supply chain systems operating under CS partnership, wherein different raw material replenishment policies are considered.

The first part of this work aims at developing what is called integrated procurement-production (IPP) system that presents a consignment partnership model in the context of a three-stage integrated forward supply chain system. The key objective of developing such models is to assess the impact of consignment stock partnership on the procurement, lot-sizing and shipping policy as well as the chain wide total cost of a three-stage forward supply chain system. Furthermore, these models will be used at a later stage, after developing the models in the second part of this thesis, to assess the impact of remanufacturing on the chain-wide total cost.

The second part of this work to be carried out in this research is based on and extends the work that has been done by Hariga et al. [24]. In their work, they developed a two-stage closed loop supply chain model in which the vendor and the buyer adopt a CS partnership, with no prior restriction on the production sequence. The extension will be conducted by integrating a third layer (supplier) along with the vendor and the buyer where the centralized decisions will now take into account the raw material related costs. A manufacturing system that procures raw materials from suppliers and converts them into finished products and at the same time processes returned products into finished products, and then ships the finished products to the buyer who has a CS partnership with the manufacturer is considered in this research work. The diagram presented in Figure 3 illustrates the three-stage closed loop supply chain system with CS partnership.

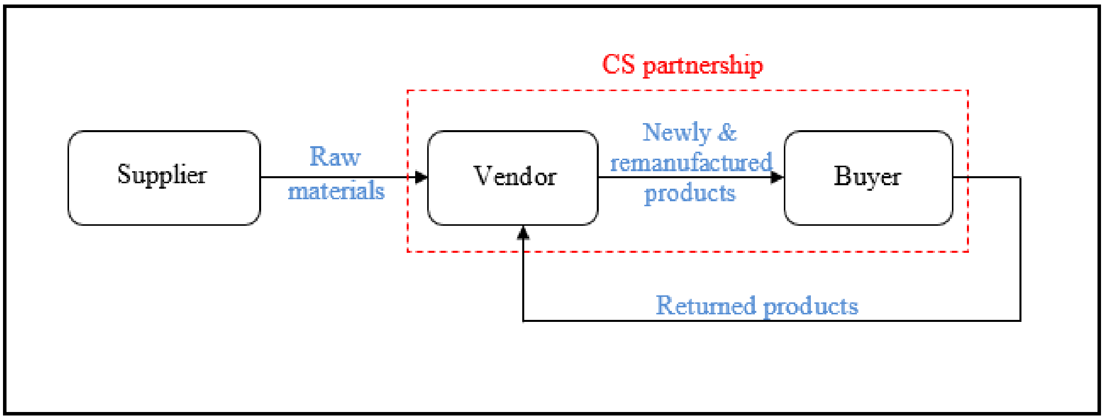

Figure 3: Three-stage closed loop supply chain system under CS partnership

The integrated procurement-production distribution problem will be formulated as mathematical model that jointly seeks to optimize:

- Procurement decisions of raw material from the upstream supplier which includes the quantity and frequency of raw material replenishments.
- The length of the production cycle.
- Number of newly and remanufactured batches produced within one production cycle. This coupled with the cycle length establishes the lot sizes of the newly and remanufactured batches.
- The sequence to follow in the production of newly and remanufactured products.
- Delivery schedule (quantity and frequency) of the finished products to the downstream buyer.
- Initial inventory level of the returned product at the vendor's stage.
- Initial inventory level of the finished products at the buyer's stage.

Hence, the objective of the models to be developed is to minimize the joint total cost comprised of holding cost of the raw material, returned product and finished product at both the vendor and the buyer stages, the vendor's setup cost associated with (re)manufacturing, and the ordering cost of the vendor and the buyer. The mathematical models will be developed under the following assumptions:

- Demand is deterministic and constant over the planning horizon (D).
- Return rate is deterministic and constant (r)
- Manufacturing and remanufacturing setup costs are different ($S_m \neq S_R$)
- Return rate is a proportion of the demand ($r = \alpha D$)
- Manufacturing (P) and remanufacturing (R) rates are finite, where $P > R$.
- Manufactured and remanufactured products are identical and have the same quality.
- Manufacturing rate is greater than demand rate ($P > D$)
- No backorders are allowed.
- Batch sizes are identical in one cycle (T) for manufactured or remanufactured products.
- Identical production for all cycles (repetitive cycles).
- There is no capacity restriction on the supplier's ability to provide necessary raw material.
- Remanufacturing rate is greater than return rate ($R > r$).

- No disposal of returned products.
- Raw materials are non-perishable, allowing for possible storage of those materials across several cycles.
- Production of one new product requires one unit of raw material item (i.e. conversion rate is equal to one).

Moreover, mathematical models will be developed for three different scenarios for procurement of raw materials:

1- **One-to-one policy:** one shipment of raw materials is enough to cover one whole cycle of production.
2- **One-to-Multi:** one shipment of raw materials is enough to cover more than one cycle of production.
3- **Multi-to-one:** multiple shipments of raw materials are required to fulfill the production of one cycle.

The following example illustrates the difference in inventory levels among the three possible scenarios. Assume a manufacturer is producing finished products through manufacturing and remanufacturing in the following sequence: (M, R, R, R, M, R, M), where Q_M and Q_R are manufactured and remanufactured lot sizes, respectively, and Q_{RM} represents: Raw material ordered lot size. Figures 4 to 7 illustrate the inventory levels of this example.

From the example, it is clearly seen that the production sequence and batch sizes during the cycle time affect the level of inventories in the buyer and vendor sides and hence affect the chain total cost. Thus, including the procurement decisions of raw material will have a great impact on changing the decisions about production sequence and quantities which in return affects the level of inventories and hence affects the total cost of the chain. As such, our proposed analysis seeks to develop a centralized policy that jointly optimizes the interrelated decisions of production sequencing, production lot sizing, raw material procurement as well as delivery schedule to the downstream retailer so that the chain wide total cost is minimized.

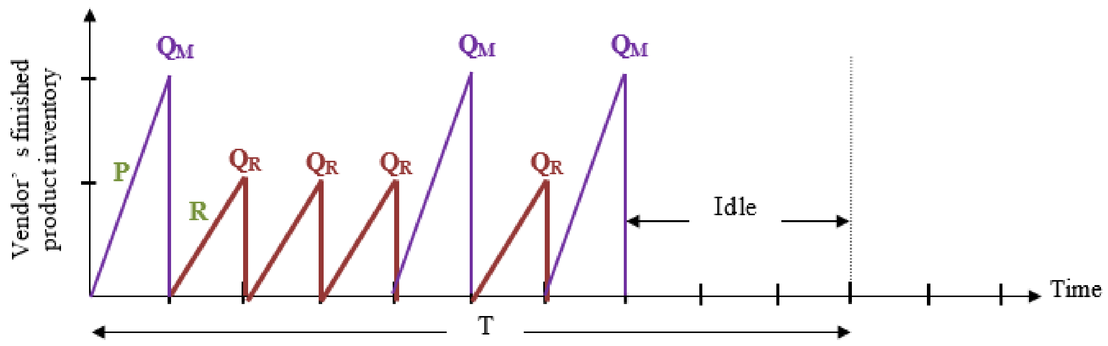

Figure 4: One possible production sequence of a product for one cycle

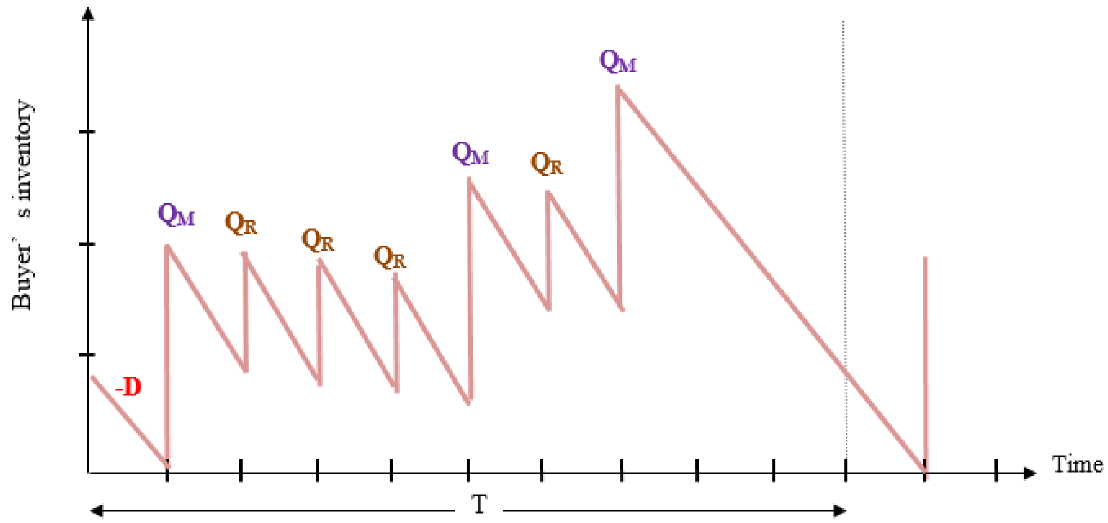

Figure 5: Buyer's inventory

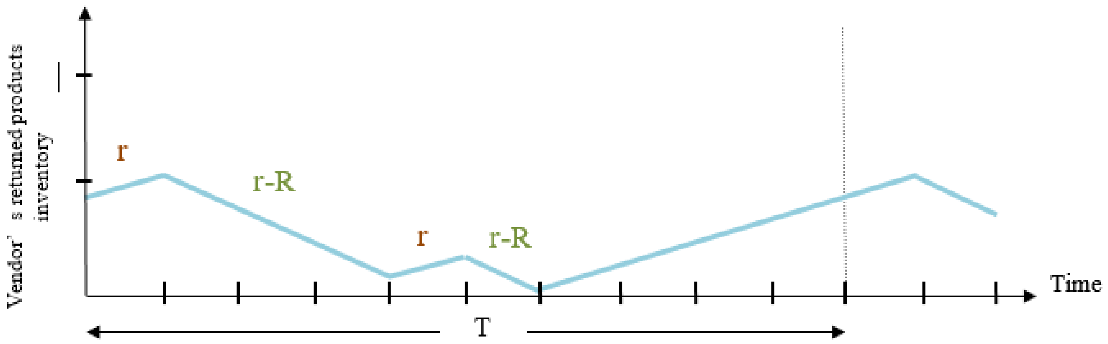

Figure 6: Vendor's returned product inventory

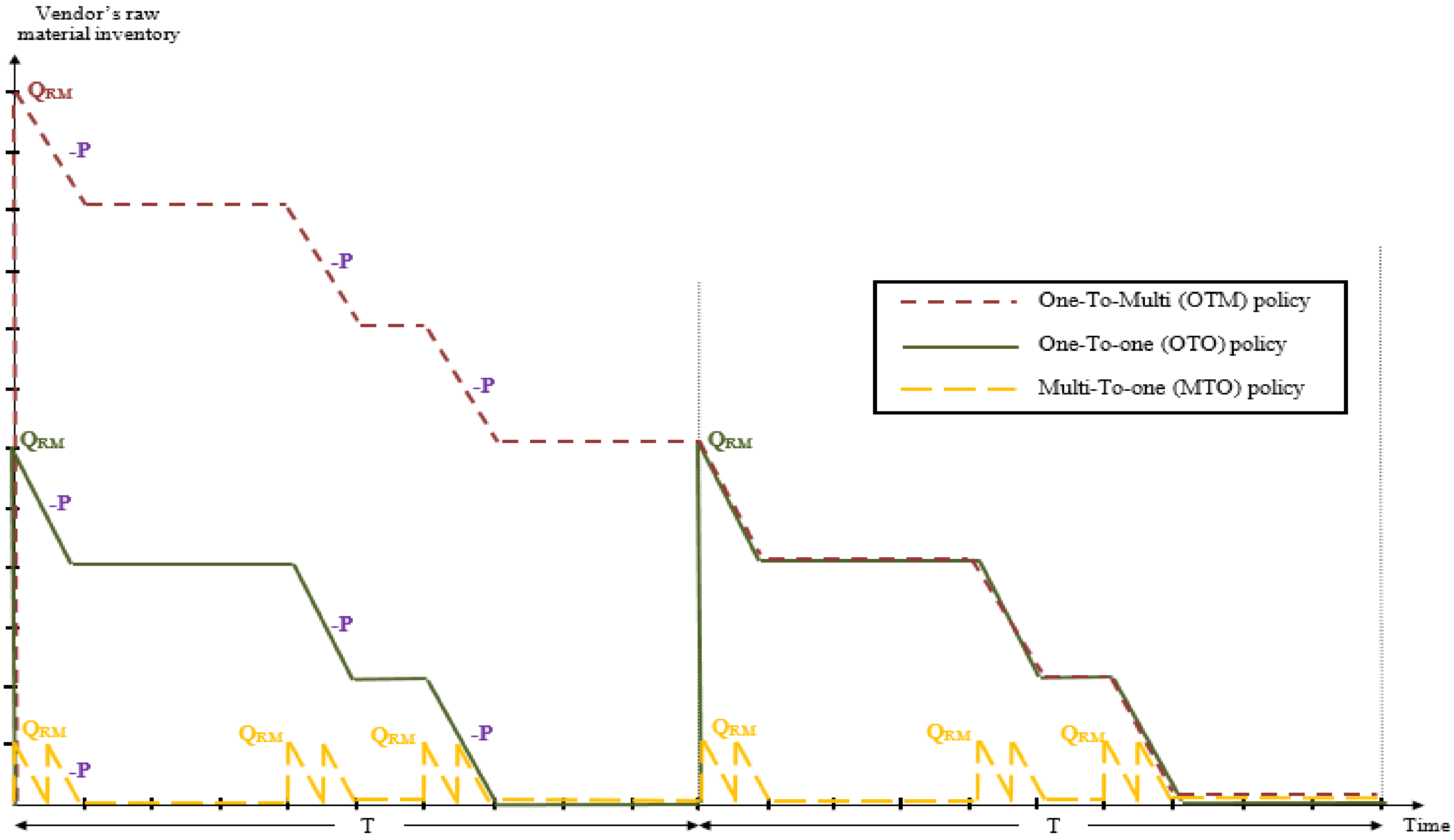

Figure 7: Vendor's raw material inventory for the three policies

From the previous example, it is clearly seen that the production sequence and batch sizes during the cycle time affect the level of inventories in the buyer and vendor sides and hence affect the chain total cost. Thus, including the procurement decisions of raw material will have a great impact on changing the decisions about production sequence and quantities which in return affects the level of inventories and hence affects the total cost of the chain. As such, our proposed analysis seeks to develop a centralized policy that jointly optimizes the interrelated decisions of production sequencing, production lot sizing, raw material procurement as well as delivery schedule to the downstream retailer so that the chain wide total cost is minimized.

1.8. Objectives and Significance

The integration of raw materials procurement decisions in the context of forward and reverse supply chain system, simultaneously, is of paramount importance as it greatly impacts the efficiency and performance of the supply chain as a whole. The significance of this research is to propose mathematical models for a three-stage forward and closed loop supply chain system operating under CS initiative, where the developed models seek to jointly optimize the centralized decisions of raw material replenishment, production sequencing and lot-sizing, and dispatching policy such that the chain wide total cost is minimized. The proposed research is the first to incorporate raw material procurement decisions for a three-stage forward and closed loop supply chain (CLSC) operating under consignment stock partnership with full flexibility of raw material replenishment and production sequence in the CLSC system. As the inventory of raw materials is depleted only once newly manufactured products are being produced and remains unchanged during the period where remanufacturing from returned items takes place, the ordering policy is directly linked to the adopted production sequence at the vendor's stage.

Those decisions, among many others, will be optimized simultaneously through the mathematical models in order to ensure the achievement of the minimum total cost for all parties in the supply chain. An extensive numerical study will also be carried out to compare the efficiency and total supply chain cost of these models and other models where a two-stage supply chain system is addressed.

1.9. Research Methodology

To achieve the objectives of this research work, the following steps will be undertaken:

Step 1: Conduct a literature review in the topics related to forward and closed loop supply chain management, CS partnership, and three stage supply chain systems.

Step 2: Develop and solve a mathematical model for a three-stage forward supply chain system operating under CS partnership with one-to-one raw material replenishment policy.

Step 3: Develop and solve a second mathematical model for a three-stage forward supply chain system operating under CS partnership with one-to-multi raw material replenishment policy.

Step 4: Develop and solve a third mathematical model for a three-stage forward supply chain system operating under CS partnership with multi-to-one raw material replenishment policy.

Step 5: Develop and solve a mathematical model for a three-stage closed loop supply chain system operating under CS partnership with one-to-one raw material replenishment policy.

Step 6: Develop and solve a second mathematical model for a three-stage closed loop supply chain system operating under CS partnership with one-to-multi raw material replenishment policy.

Step 7: Develop and solve a third mathematical model for a three-stage closed loop supply chain system operating under CS partnership with multi-to-one raw material replenishment policy.

Step 8: Perform sensitivity analysis in order to determine the impact of key problem parameters on the performance of the developed models.

Step 9: Compare the performance of the developed models, with each other, as well as with other models under different settings of the problem parameters.

Chapter 2: Literature Review

In this chapter, a review of the literature related to the problem addressed in this research will be presented. The chapter starts by reviewing the Joint Economic Lot Sizing (JELS) that is considered as the base of integrated supply chain models. After that, it will review the main topics that this research is addressing, namely consignment stock, closed loop supply chain as well as raw material replenishment decisions in the context of multi echelon supply chain systems.

2.1. Joint Economic Lot Sizing (JELS)

In integrated supply chain systems seeking to jointly minimize the costs of all parties involved, JELS is considered to be the building block of such integration. According to Banarjee [25], economic benefits to both parties, vendor and buyer, can be achieved by adopting a joint policy through a spirit of cooperation between them. In JELS models, multiple product batches are manufactured by the vendor and shipped to single or multiple buyers. Hence, as an effort to reduce the overall costs of the whole system (the integrated supply chain), attempts were made to jointly determine the optimum batch sizes as well as number of shipments.

One of the first models of JELS was developed by Goyal [26]. In his model, he assumed a deterministic and constant demand and an infinite production rate for a two-stage supply chain system comprised of a single vendor and a single buyer. As a result of his study, he found that production of variable sub-batches sizes (unequal sizes) reduces the annual variable costs and the cycle time, which results in reducing total costs. Later, similar models were developed by Banarjee [25] and Monahan [27] assuming a finite production rate. Monahan [27] proposed adjusting the present pricing schedule of the vendor to be used as an incentive to increase the quantity ordered by the buyer and developed an optimal quantity discount schedule. This cooperation benefits both, the vendor and the buyer and results in larger but less frequent orders which benefit the vendor in reducing transportation costs, in-processing orders and set up costs. It also benefits the buyer in receiving lower prices for larger order quantities. Banarjee [25] extended Monahan’s work by including the inventory carrying cost of the vendor in his formulation.

Later, Goyal [28] proposed a model in which the production lot size is dispatched to the buyer in a number of equal sized shipments where no shipping can take place prior to completing the production of the whole lot. Lu [29] extended this work by relaxing the assumption that was made by Goyal and found optimal solutions where shipping can take place before completing the production of the batches. In his model, he considered two different scenarios, single vendor - single buyer and single vendor - multiple buyers. Building on Lu's idea, Goyal [30] showed that within a production batch increasing the shipments sizes by a fixed factor leads to a solution with lower cost.

Hill [31] contributed to this field by optimizing the number of deliveries for a single vendor – single buyer problem by taking into consideration the production set up costs, costs of delivery and inventory holding costs of both parties. In his model, he showed that the model results in a lower cost solution and the results are very sensitive to the holding cost of the vendor and the buyer. In case their holding costs are similar, then Goyal's model of increasing successive shipment size is better than the model of equal shipment sizes developed by Lu [29]. A comprehensive review of the contributions pertaining to the JELS is presented by Glock [32].

2.2. Consignment Stock

The JELS models have been extended to account for situations involving CS agreement. Those developed models of CS agreement dealt with situations where a single vendor is coordinating with single buyer or multiple buyers.

Valentini and Zavanella [33] developed a model that takes into account the holding costs of the inventory, capital (opportunity) and storage costs. In their analysis, they did not consider the ordering cost. According to their findings, having a CS agreement between the vendor and the buyer results in better system efficiency and more benefits than the independent classical inventory models. In addition to that, the implications on companies that undergo CS agreement were analyzed by the authors. They showed that the buyers benefit from this agreement by transferring the capital cost to vendors by paying them only when they use the items as well as reducing the stock out issues. Moreover, the vendor benefits from this agreement by transferring the storage cost to the buyer. However, in such agreement, efficient information sharing is crucial in maintaining successful CS partnership.

Braglia and Zavanella [34] showed in their single-vendor and single-buyer model that in environment where lead times and demand are not constant but varying over time, it would be very advantageous to have CS partnership, and they stated that “CS policy might be a strategic and profitable approach to stock management in uncertain environments” [34]. CS would be beneficial in varying demand environment for the fact that the vendor will guarantees for the buyer minimum and maximum stock levels so that the buyer can satisfy the varying demand without any stock outs.

Persona, Grassi, and Catena [35] analyzed in their models the different policies of transportation in a single-vendor/ multiple buyers system. The results showed that there is dependency between the overall cost savings and cost parameters, demand, and production schedules. In their CS models, the authors also considered the case of items’ deterioration resulting in lower overall cost savings with higher variability of production.

In another work, Gumus, Jewkes, and Bookbinder [36] examined CS partnership in an environment where demand is deterministic. They determined the operational benefits beyond having CS partnership in such environment. Their model included the vendor’s holding, releasing and setup costs and the buyer’s carrying and ordering costs. They showed that both the vendor and the buyer benefit from this agreement. From the buyer side, benefits are realized due to the payment of the vendor after selling the product while the vendor may benefit by achieving savings through reducing the opportunity cost carried by him. However, in case that opportunity cost could not be reduced and the only benefiting party is the buyer, then incentives such as wholesale price increment could be used to convince the vendor to undergo such agreement.

One of the recent contributions in CS was done by Hariga et al. [37]. They developed a model to determine the optimal production lot sizes and the optimal delivery schedule to buyers in a single-vendor/ multiple buyers system. They assumed a finite production rate and a deterministic demand in solving their model. They developed a non-linear mixed integer programming formulation for the problem and then built on it to develop a near-optimal delivery schedule using a heuristic procedure.

2.3. Closed Loop Supply Chain

Closed loop supply chain (CLSC) management has gained the attention of many firms nowadays in an attempt to become environmental friendly and to realize benefits from the recovery of their manufactured products after being used by the end customers. Many studies have been conducted in this field laying the ground for firms to have a closed loop supply chain systems. The models are developed in single and multi-stage systems for combined forward and reverse supply chains.

In optimizing the forward supply chain, EOQ and/or EPQ models could be used to determine the optimum quantities to be ordered and/or manufactured. This decision is based on having the optimum balance between the ordering/setup cost and the holding cost of finished products. However, in CLCS the process is more complex where we will have to include another tradeoff concerning the use of the valuable returned items for remanufacturing or to manufacture using newly procured items. Moreover, a decision has to be made for the returned items whether to dispose them or store them in special inventories for remanufacturing.

Based on the joint EPQ and EOQ, Koh et al. [38] developed models to determine the optimal inventory levels of procured and recoverable items simultaneously. The model considers finite repair capacity and the time needed to transform recoverable into serviceable products. The return rate of recoverable products was assumed to be fixed besides deterministic demand, repair capacity, lead times, and recovery rate, and the assumption that procurement is less cost effective than repairing. Moreover, they added another assumption that if multiple recovery setups are made then only one procured item order can be made, or if multiple procured item orders are made then only on recover setup can be made. This assumption is a limitation for their model since in reality multiple setups and orders can be made simultaneously. Under the stated assumptions, they found the optimal number of recovery setups for one procured order or the optimal number of procured orders for one recovery setup.

Minner and Lindner [39] developed a closed loop model under the assumptions of deterministic demand and return rate and identical batch sizes. They determined the optimal batch sizes for the newly produced and recovered items for two scenarios. In the first scenario, they considered consecutive production of newly produced items first followed by identical remanufactured batches. In the second scenario, they considered a reverse scenario where the recovered products are remanufactured first followed by identical batches of newly produced products. They concluded that if the reuse rate is high and approaches 1, then the second scenario is to be used. On the contrary, for low values of the reuse rate then the first scenario is to be used. Following this, the assumption of identical batches has been relaxed by the authors. Following this relaxation, they found that the final batch size of the remanufactured products should be smaller than the previous batches.

Another contribution to green supply chain field was done by Choi et al. [40]. Their model determined the optimal quantities for procured and recovered items simultaneously. The model was developed under the assumptions of equal lot sizes for procured and recovered items and deterministic demand and recovery rates, where the demand rate is greater than that of the recovery. By analyzing the model and using sequencing algorithms, the optimum sequence of set up and order was determined, and accordingly inventory levels, number of setups, optimal procurement orders and total cost were calculated.

Researchers such as Chung et al. [41] and Mitra [42] have developed models for forward and reverse supply chain in multi echelon systems. Chung et al. studied a model where reconditioning facilities are recovering the returned products and then return them back to retailers after being reprocessed. They found that integrated policies combining a third party recycle dealer along with single supplier, manufacturer and retailer (centralized system) yields to greater combined profits than a decentralized system. Mitra [42] presented a model where distributers place orders to depot. In his model, serviceable inventory shortage costs, inventory and setup costs at each level along with deterministic demand were considered. Combining all together, his model was able to determine the optimum number of cycles and ordering quantities at the distributer level.

Recently, Jaber et al. [43] developed a model for a two-stage closed loop supply chain comprised of a single vendor and a single buyer. They assumed that the vendor and the buyer adopt a consignment stock policy. Under this assumption, they developed a mathematical model for the case where the newly and remanufactured batches are of equal and different sizes. Moreover, they took into account the costs of transportation, sorting and inspection in their model. However, they assumed that the production sequence is known a priori where a number of remanufactured batches are produced first followed by consecutive new batches. By analyzing the results, they found that the batch sizes and total cost of the system are significantly affected by the collection rate of the repairable and used items. They suggested that operating such a system at its maximum environmental edge may not be possible without tax brackets and incentives. However, the system may undergo a continuous development and improvement process which will reduce its operational costs to make it possible to operate at its maximum environmental edge but additional investment may be required.

Hariga et al. [24] developed a two-stage closed loop supply chain model for a single-vendor single-buyer. In their model, the vendor and buyer implement CS partnership. The difference between this later work and that of Jaber et al. [43] is in the sequencing of the newly and remanufactured batches where the production sequence in Hariga's work is optimized through the solution of the developed mathematical model rather than taken as an input. The developed model also optimizes the initial inventory levels of the finished products and the returned items at the buyer's and vendor's stages, respectively. An iterative solution algorithm that efficiently attains near-optimal solutions was also proposed.

Unlike other researchers who developed models that enforce equal lot sizes with a specific cycle structure over the planning horizon, Schulz and Voigt [44] developed a flexibly structured heuristic model where remanufacturing batch sizes are allowed to have different sizes over the planning horizon. They assumed a constant demand, and only a fraction of the demand is returned to the manufacturer for remanufacturing, where it is kept in inventory until needed. In their results, they determined total cost of their flexible model and compare it to the models of equally sized batches. They showed that their flexible approach outperforms the equally sized batches of existing approaches by up to 17% in more than half of all instances.

Govindan et al. [45] combined and reviewed a total of 382 recently published papers in closed loop supply chain and reverse logistics. Those papers are analyzed and categorized so that a useful foundation of past research is constructed. Moreover, they identified the gaps in literature and further suggested opportunities for future research.

2.4. Integration of Raw Material Procurement Decisions

Since production of finished products requires the use of raw materials, the procurement costs associated with it have an impact on total cost of the overall system. Hence, a three-stage supply chain system that includes suppliers, manufacturers, and retailers, should be analyzed in order to come up with the production and shipping policy minimizing the chain wide total cost.

Sarker and Khan [46] developed a model for a single stage supply chain. Their model proposes a raw material ordering policy that will meet the requirements of a production facility to satisfy its demand. The manufacturer receives raw material from suppliers and delivers the finished products to the buyers. First, they developed a general cost model considering raw materials and finished products and then used this model to determine the optimal manufacturing batch sizes and ordering policy of raw materials that will minimize total cost of the system. Their model is restricted to the case where the delivery of the finished product cannot take place until the whole lot is produced with complete quality certification. In order to simplify their model, three scenarios have been considered; ordering a quantity of raw materials that is required to produce one lot of finished product, ordering a quantity of raw materials required to produce multiple lots of finished product, or multiple orders can take place to produce one lot of finished product. In addition, they analyzed each scenario with two different delivery cases: continuous or periodic (single or multiple installments). According to their analysis, they showed that their model results in lower total system costs compared to disjoint systems that operate under separate policies.

Parija and Sarker [47] developed a forward supply chain model to determine the optimal production batch size and the optimal multi-ordering policy of raw materials that will minimize the finished goods and raw material inventory costs as well as the ordering cost. The system consists of a manufacturer who supplies multiple customers with fixed quantities of finished goods at fixed interval deliveries. Moreover, the model determines the optimal starting time of each batch to benefit from the carried-over inventory from one cycle to another. In developing the model, they assumed deterministic and fixed demand, non-perishable raw material is used, instantaneous supply of raw material to the manufacturer, and production is always sufficient to fulfill the demand (i.e. no finished goods shortages). They found that the total cost of the integrated production inventory system can be minimized through the optimization of the manufacturing batch size and the procurement of its raw materials.

Lee [23] considered a single forward three stage supply chain system where a single manufacturer is ordering raw materials from a single supplier and delivers the finished goods after processing to a single buyer. He assumed a finite production rate of lot sizes, periodical deliveries of the finished goods, fixed lot sizes, and constant demand rate. He developed an integrated model to minimize the total supply chain cost per unit time comprised of ordering and holding of raw materials, finished goods holding and production setup cost of manufacturer, and inventory holding and ordering cost of the buyer. His model determines the optimal lot size of raw material deliveries to the manufacturer, production batch size, and buyer's ordering schedule. The analysis and numerical examples showed that the integrated system results in less mean total cost than the systems that does not jointly consider the inventory costs.

As discussed in CLSC literature review, the model of Chung et al. [41] considered a closed loop three-stage supply chain where the supplier is part of the wide chain. Their model meant to maximize the joint profit among all the parties from the supplier to the retailer through optimizing the replenishment and production strategies. They assumed in their model infinite planning horizon, no stock outs and deterministic lead times, demand and return rate as well as remanufacturing and manufacturing rates. The results showed that it is more profitable to have an integrated system, around 3.67% higher, rather than operating individually.

Jaber and Goyal [48] developed a forward three-stage supply chain model consisting of multiple suppliers, a vendor, and multiple buyers. The decision process of the entire chain is centralized where the developed model guarantees for all parties to either realize a decrease in the local cost due to coordination or have it the same as the individual optimized policy. This guaranteed that savings is assumed to be distributed and shared among all the parties included. The model has been developed based on deterministic demand assumption and all non-identical buyers have a common cycle time. The results showed that this coordination results in around 40% reduction in holding and ordering costs. However, some of the parties, such as the buyer, will have increased costs while others may benefit from coordination. Hence, to overcome the losses that may occur to some parties, a quantity discount form supply chain should be used to compensate the losses.

Sarker and Diponegoro [49] developed an optimal policy for procurement and production in a forward three stage supply chain system that consists of multiple suppliers, a manufacturer, and multiple buyers. The model has been developed to determine the optimal initial and ending inventory, production start time, raw material replenishment orders in each cycle, the cycle length as well as the number of cycles so that total cost of the wide chain system is minimized. They assumed non-competing and non-identical suppliers and buyers, respectively.

Unlike other existing research works that focus on a single vendor-buyer supply chain, Ben-Daya et al. [20] developed a three-stage supply chain model with JELP. The model consists of a supplier, a manufacturer, and multi-retailers. The main objective of this study is to minimize the chain-wide total cost including raw material inventory holding cost, ordering and setup costs, and inventory holding cost of finished products. This objective is achieved through the developed model by determining the quantities and timings of outbound and inbound logistics for all the parties across the chain. In developing the model, they assumed that the cycle time at each stage is an integer multiple of the adjacent downstream stage, and allowed for shipments from a particular lot to take place before producing the entire lot. In order to derive near-optimal solutions for the model, derivative-free methods were used. By comparing the developed model with previous models that do not consider the optimization of raw material replenishment, they showed that their model outperforms the previous models for most of the problem instances and results in lower total supply chain cost.

Kundu and Chakrabarti [50] developed an integrated multi-stage supply chain inventory model that seeks to minimize the chain-wide total cost by coordinating the replenishment decisions of raw material procurement and finished products production and delivery activities. The developed model assumes an imperfect production system in which defective items are produced randomly and are assumed to be re-workable.

The supply chain in their model comprise of multiple suppliers, single manufacturer and multiple buyers. In their work, the quantities of raw material orders are assumed to be fixed, however, the manufacturer delivers the finished goods in unequal shipments to each buyer. They developed a simple algorithm to obtain an optimal production policy that seeks to minimize the expected average chain-wide total cost of the integrated production-inventory system.

Unlike other researchers who addressed centralized systems, Taleizadeh and Noori-Daryan [51] presented a decentralized three-layer supply chain system that consists of a single supplier, single vendor, and multiple retailers in which orders are placed from the retailers to the vendor to satisfy their demand, and the vendor places raw material orders to the suppliers to produce the products and fulfil the retailers' orders. In their work, demand is assumed to be price sensitive and shortages are not allowed. The objective of this work is to minimize the chain-wide total cost by coordinating decision-making policy using Stackelberg–Nash equilibrium. The decision variables in their model are the prices of raw materials and finished products as well as the number of raw material and finished products shipments.

The case study conducted by Yan et al. [52] examined the green component procurement collaborations and their benefits in terms of improved shipping time performance and shared costs. Statistical validations with empirical data and system dynamics simulation analysis have been used. The results obtained from their work suggested that cost effectiveness and shipping time efficiency are significantly affected by the collaborative planning practices for procurement quantities and accurate fulfillment by the suppliers. They also found that it is significantly advantageous for the supply chain collaboration to have a shared cost reduction for procurement of components.

Tables 3 and 4 provide a classification of the most relevant literature from the view point of several dimensions for the forward and closed loop supply chain, respectively, which better helps in positioning and benchmarking the work presented in this thesis against those available in the literature. The work done in this thesis makes several contributions. Firstly, it is the first to address and propose mathematical models for a three-stage forward and closed loop supply chain systems operating under consignment stock (CS) partnership in which the inbound and outbound logistical decisions are simultaneously accounted for, coupled with the production sequencing decisions at the vendor stage. Moreover, it is the first to incorporate raw material procurement decisions with no prior restriction on the raw material replenishment policy or the production sequence in the CLSC system. More specifically, it is the first to formulate mixed integer non-linear programming (MINLP) models that jointly seek to optimize the procurement decisions of raw material, the length of the production cycle, the sequence to follow in the production of newly and remanufactured batches, number of newly and remanufactured batches produced within one production cycle, as well as the initial inventory levels of recovered and finished products at the vendor's and buyer's premises, respectively, in order to minimize the chain-wide total cost per unit time.

Table 3: A classification of the relevant literature (forward supply chain)

Paper	Supply chain configuration	Consignment partnership	Integration of raw material procurement	Market demand		Production rate	
				Deterministic	Stochastic	Finite	Infinite
Goyal, S. K. [26]	SV-SB[a]			x			x
Monahan, J. P. [27]	SV-SB			x		x	
Banerjee, A. [25]	SV-SB			x		x	
Lu, L. [29]	SV-SB/MB[b]			x		x	
Goyal, S. K. [30]	SV-MB			x		x	
Hill, R. [31]	SV-SB			x		x	
Valentini, G., & Zavanella, L. [33]	SV-SB	x			x	x	
Braglia, M., & Zavanella, L. [34]	SV-SB	x		x	x	x	
Persona et al. [35]	SV-MB	x			x	x	
Gumus et al. [36]	SV-SB	x		x		x	
Hariga et al. [37]	SV-MB	x		x		x	
Sarker, R. A., & Khan, L. R. [46]	Single echelon		x	x		x	
Parija, G. & Sarker, B. [47]	SS-SV-MB[c]		x	x		x	
Lee, W. [23]	SS-SV-SB		x	x		x	
Jaber, M. Y., & Goyal, S. K. [48]	MS-SV-MB[d]		x	x		x	
Sarker, B & Diponegoro, A. [49]	MS-SV-MB		x		x	x	
Ben-Daya et al. [20]	SS-SV-MB		x	x		x	
Kundu, S., & Chakrabarti, T. [50]	MS-SV-MB		x	x		x	
Taleizadeh & Noori-daryan [51]	SS-SV-MB		x		x	x	
Yan et al. [52]	MS-SV-SB		x	x		x	
This Thesis	**SS-SV-SB**	**x**	**x**	**x**		**x**	

[a] Single Vendor-Single Buyer, [b] Single vendor-multiple Buyers, [c] Single supplier-vendor multiple buyers, [d] Multiple suppliers-single vendor-multiple buyers

Table 4: A classification of the relevant literature (closed loop supply chain)

Paper	Supply chain configuration	Batch sequencing	Consignment partnership	Integration of raw material procurement	Market demand		Production rate	
					Deterministic	Stochastic	Finite	Infinite
Koh et al. [38]	Single echelon	(M,1) or (1,R)			x		x	x
Minner & Lindner [39]	Single echelon	General			x			x
Choi et al. [40]	Single echelon	General			x		x	x
Mitra, S. [42]	SV-SB	Simultaneous			x	x	x	x
Schulz & Voigt [44]	Single echelon	General			x			x
Jaber et al. [43]	SV-SB	(R,M)	x		x		x	
Hariga et al. [24]	SV-SB	General	x		x		x	
Chung et al. [41]	SS-SV-SB	(R,M)		x	x		x	
This Thesis	**SS-SV-SB**	**General**	**x**	**x**	**x**		**x**	

Chapter 3: Three-Stage Forward SC with CS Partnership (3S-FSC-CS)

Being a form of strategic partnership that poses several potential benefits to members of the supply chain, this chapter presents a consignment partnership model in the context of a three-stage centralized supply chain system. Although the model presented in this chapter addresses the single-vendor single-buyer problem, it is referred to as a three stage since the raw material procurement decisions at the vendor stage is also optimized alongside the lot sizing and shipment decisions of the finished product. As pointed out by Lee [23], models that jointly account for raw material procurement as well as manufacturing setup are referred to as integrated procurement-production (IPP) systems. To the best of our knowledge, consignment stock partnership has not been addressed in the context of such systems before.

In order to assess the impact of consignment stock partnership on the procurement, lot-sizing and shipping policy as well as the chain wide total cost, a centralized supply chain comprised of a single vendor/manufacturer and a single buyer operating under CS partnership is analyzed. The manufacturer is only producing newly manufactured products at each production run and orders a raw material quantity in lots of size Q_{rm} each according to three raw materials ordering policies: One-To-One (OTO), One-To-Multi (OTM) and Multi-To-One (MTO). In OTO policy, the vendor orders one raw material shipment of size Q_{rm} that is enough to cover the production for one whole cycle where Q_{rm} is ordered and received at the beginning of each cycle. However, in OTM policy, the vendor orders one raw material shipment of size Q_{rm} that is enough to cover the production for multiple cycles. In this policy, Q_{rm} is ordered and received at the beginning of a cycle. Finally, when operating under MTO policy, the vendor orders multiple raw material shipments of equal size (Q_{rm}) each during one production cycle to satisfy the production of the finished product in that cycle. In this latter policy, each raw material shipment is enough to produce an amount that satisfies "m" fraction of the demand in one cycle. For instance, if $m = 0.5$ this means that Q_{rm} is enough to produce 50% of the demand in one cycle implying that we have to order twice in once cycle in order to fully satisfy the demand. Note that, under this policy, Q_{rm} is ordered and received when the inventory level drops to zero. It can be observed from the vendor's raw material inventory profile shown in Figure 8c that different ordering policies obviously affect the levels of inventory and hence inventory holding cost. In order to strike a balance between the ordering cost and the inventory holding cost, the effect of raw material ordering policy will have to be explicitly accounted for in the optimization problem at hand.

When operating under CS partnership, the vendor ships the manufactured batches as soon as they are produced in order to minimize its inventory holding cost of finished products. During the production uptime portion of the cycle, the vendor ships equal batches of size Q each, and the time taken to produce one batch of size Q is called the production time per lot (T_P). The lot size, Q, and the production time per lot can be expressed in terms of the cycle time (T) and number of production batches (n) as follows:

$$Q = \frac{D\,T}{n} = qT \tag{1}$$

$$T_P = \frac{Q}{P} = \frac{\frac{D\,T}{n}}{P} = \frac{D}{P\,n}T = t\,T \tag{2}$$

where, D is the demand rate and P is the production rate. In addition, the total production uptime can be determined as a function of the production time per lot (T_P) and number of production runs (n) and is calculated as follows:

$$nT_P = n\left(\frac{D}{P\,n}T\right) = \frac{D}{P}T \tag{3}$$

Accordingly, the idle time is:

$$T - \frac{D}{P}T = T\left(1 - \frac{D}{P}\right) = T\rho \tag{4}$$

Sub-figures a, b and c in Figure 8 show the vendor's raw material and finished product inventory profile as well as the buyer's finished product inventory profile for the case where the vendor has 5 production runs ($n = 5$).

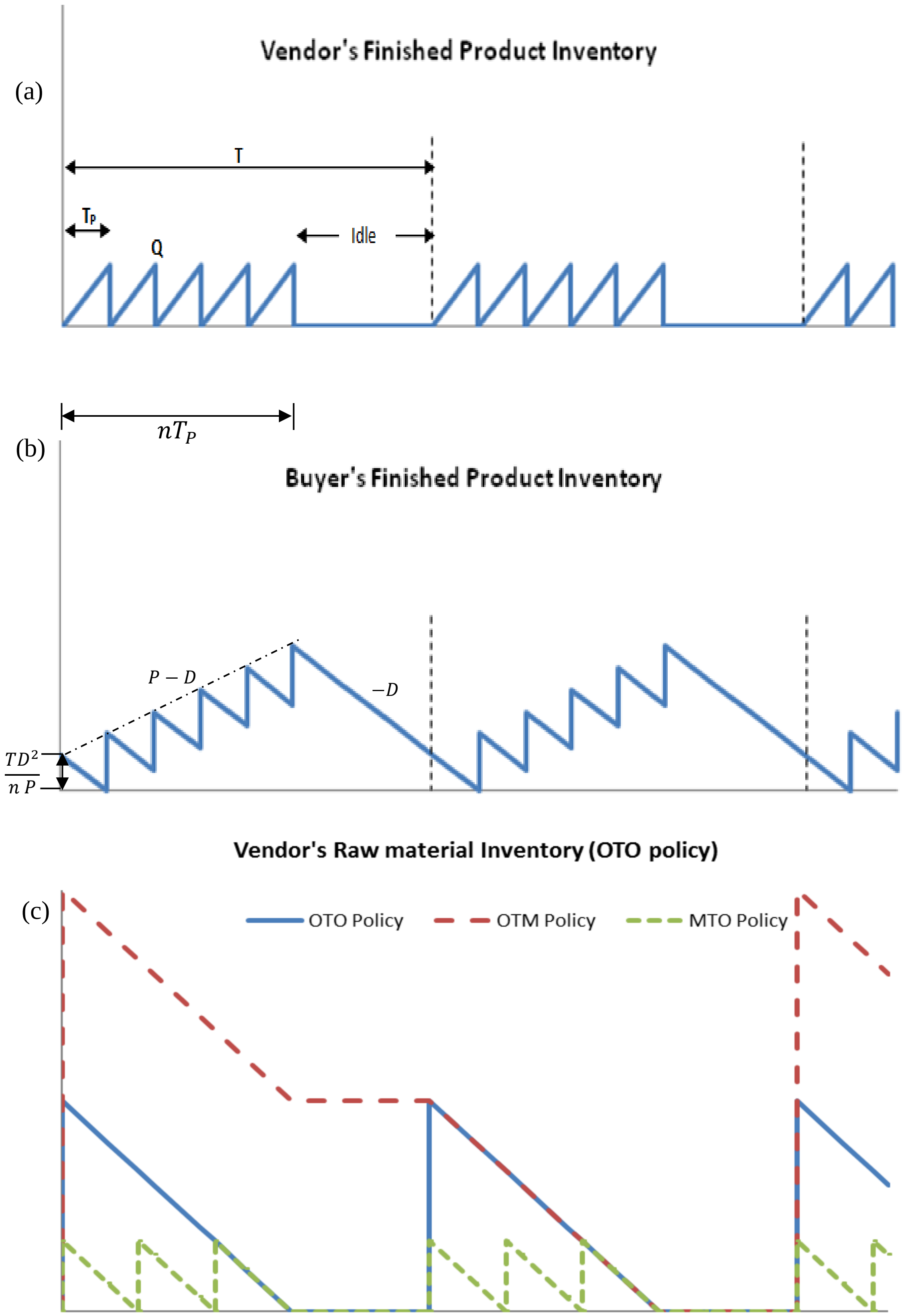

Figure 8: Inventory levels of the system for n=5

The objective of the integrated model is to determine the optimal production batch size (Q) and number of shipments (n) that will minimize the total long-run system wide cost consisting of the inventory holding costs of raw materials and finished products at the vendor's stage, finished products holding cost at the buyer's stage as well as the buyer's and vendor's ordering costs, and the vendor's production setup cost. The following sub-sections provide the derivation of these cost components under the three raw material replenishment policies explained earlier.

3.1. Derivation of The Vendor's Finished Product Inventory Holding and Setup Costs

Referring to Figure 8a, we can observe that the vendor's inventory profile follows that of the basic EOQ model during the production uptime of the cycle, since equal batches of size Q are produced and each batch is dispatched to the buyer every T_P units of time until production ceases. Hence, the vendor's inventory holding cost per unit of time is given by:

$$H_v = \frac{h_v}{T}\left(\frac{1}{2}T_P Q n\right) = \frac{h_v}{T}\left(\frac{1}{2}\left(\frac{D}{P\,n}T\right)\left(\frac{D\,T}{n}\right)n\right) = T.\frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) = T.RH_v \qquad (5)$$

where RH_v is the relative vendor's finished product inventory holding cost per unit of time.

Since the vendor is producing a single product on a continuous basis during the production uptime of the cycle, the setup of the production process takes place only for the first batch at the beginning of the cycle. The vendor's total set up cost per unit time is thus given by:

$$S_v = \frac{1}{T}S \qquad (6)$$

where S is the set up cost for the product.

3.2. Derivation of The Buyer's Holding and Ordering Costs

In order to determine the holding cost per unit time for the finished products at the buyer's end, we need to find the total area under the inventory graph in Figure 8b. The classical approach to determine this area is to subtract the vendor's finished products inventory from the system inventory of the finished products (vendor's and buyer's inventories). At the beginning of the production cycle, the vendor has no stock of finished product and the system inventory is at its minimum. This minimum level corresponds to the buyer's initial inventory that is enough to satisfy the demand during T_p and is calculated as follow:

$$D.T_P = D.\left(\frac{D}{P\,n}T\right) = \frac{TD^2}{n\,P}$$

In addition, the system's inventory builds up at a net rate of $(P - D)$ until it reaches its maximum level just at the time when production ceases. The maximum level the system inventory reaches is:

$$\frac{TD^2}{n\,P} + (P-D)(nT_P) = \frac{TD^2}{n\,P} + T\left(\frac{(P-D)D}{P}\right)$$

When the production ceases, the system's inventory equals to the buyer's inventory, and from this point onwards the inventory drops at a rate of (D) to reach the minimum level again at the end of the cycle. The inventory of the system (I_s) is thus given by:

$$I_s = \left[\frac{TD^2}{n\,P} + T\left(\frac{(P-D)D}{2P}\right)\right](nT_P + T\rho)$$

$$= \left[\frac{TD^2}{n\,P} + T\left(\frac{(P-D)D}{2P}\right)\right]\mathrm{T}$$

$$= T^2\left[\frac{D^2}{n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]$$

The buyer's inventory (I_b) is simply the system's inventory minus the vendor's inventory which gives:

$$I_b = T^2\left[\frac{D^2}{n\,P} + \left(\frac{(P-D)D}{2P}\right)\right] - \frac{T^2D^2}{2\,P\,n}$$

$$= T^2\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]$$

The buyer's finished products inventory holding cost per unit of time is given by:

$$H_b = \frac{h_b}{T}\left[T^2\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right]$$

$$= T.h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]$$

$$= \mathrm{T}\,.\,\mathrm{R}H_b \tag{7}$$

where $\mathrm{R}H_b$ is the relative buyer's finished products inventory holding cost per unit of time.

The buyer's ordering cost per unit of time is given by:

$$A_b = \frac{nO_b}{T} \tag{8}$$

where O_b is the buyer's ordering cost per order.

3.3. Derivation of The Vendor's Raw Material Holding and Ordering Costs

While the cost components calculated above are independent of the raw material replenishment policy adopted, the vendor's ordering and raw material holding costs are policy dependent. The latter cost can be determined by the area under the inventory profile of raw material. The level of raw material inventory always decreases by a rate of P per unit of time during the manufacturing uptime and stays constant during the idle time portion of the cycle. On the other hand, the level of inventory increases by Q_{rm} only when ordering a raw material from the supplier. As mentioned earlier, the different ordering policies affect the levels of inventory and hence inventory holding cost. Therefore, a model will be derived for each policy.

3.3.1. **One-To-One policy (OTO).** In this policy, the vendor orders one raw material shipment of size Q_{rm} that is enough to cover one whole cycle of production, in which case $Q_{rm} = DT$. This shipment is ordered and received at the beginning of the cycle. Figure 9 shows the raw material inventory profile under this policy for the same example that was mentioned earlier.

It is evident that the maximum level of inventory is Q_{rm} which takes place at the beginning of the cycle. This level decreases by a rate of $P$and reaches zero at the time when production ceases. Thus, the vendor's inventory is simply the area under the triangle which is given by:

$$\frac{1}{2}\,.\,nT_P.\,Q_{rm} = \frac{1}{2}\left(\frac{D}{P}T\right)(DT) = \frac{D^2T^2}{2P}$$

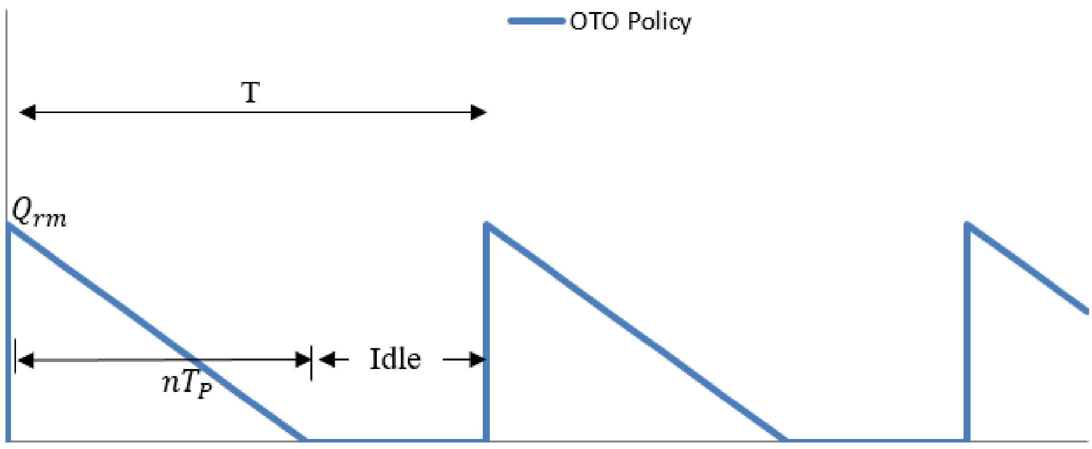

Figure 9: Vendor's raw material inventory profile under OTO policy

The vendor’s raw materials inventory holding cost per unit of time is given by:

$$H_{rm} = \frac{h_{rm}}{T}\left(\frac{D^2T^2}{2P}\right) = T.h_{rm}\left(\frac{D^2}{2P}\right) = T.RH_{rm} \tag{9}$$

where RH_{rm} is the relative vendor’s raw materials inventory holding cost per unit of time.

The vendor’s raw materials ordering cost per unit of time is simply given by:

$$A_v = \frac{O_v}{T} \tag{10}$$

where O_v is the vendor’s ordering cost of raw materials per order.

3.3.2. **One-To-Multi policy (OTM).** In this policy, the vendor orders one raw material shipment of size Q_{rm} that is enough to cover multiple cycles (u) of production. The shipment is ordered and received at the beginning of the cycle every other u cycles. This policy pays off in situations where the ordering cost is quite high, in which less frequent orders of larger batches are more cost effective. Figure 10 depicts the raw material inventory profile under this policy for the same example that was mentioned earlier where the raw material inventory is replenished once every other cycle ($u = 2$).

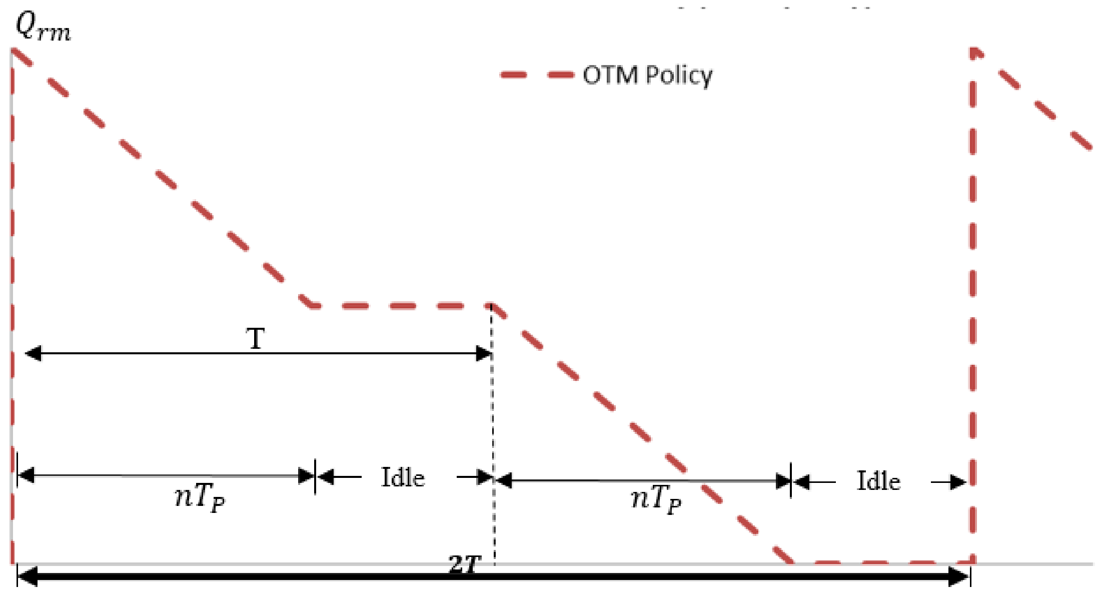

Figure 10: Vendor's raw material inventory profile under OTM policy

It can be seen from Figure 10 that the maximum level of inventory is Q_{rm}, which it is enough to satisfy 2 cycles in this example ($Q_{rm} = 2DT$). This level decreases by a constant rate of P during the production uptime portion of the cycle and remains constant by the time when production ceases (i.e. during the idle time portion of the cycle). Additionally, since Q_{rm} is enough to cover two consecutive cycles, the level of inventory when production ceases in the first cycle is half Q_{rm} (or simply DT). This level is exactly equal to the inventory level at the beginning of the second cycle. In general, the raw material inventory level drops by DT after the production uptime in every cycle until it reaches zero during the idle time of the last cycle. Thus, the inventory is calculated as follows:

$$I_{rm} = \underbrace{\frac{1}{2}(DT)nT_P + \mathrm{DT.T}}_{1^{st}cycle} + \underbrace{\frac{1}{2}(DT)nT_P}_{2^{nd}cycle}$$

$$= (DT)nT_P + \mathrm{DT}^2$$

When we order raw materials amount that covers the production for 3 cycles ($u = 3$), then

$$I_{rm} = \underbrace{\frac{1}{2}(DT)nT_P + \mathrm{2DT.T}}_{1^{st}cycle} + \underbrace{\frac{1}{2}(DT)nT_P + \mathrm{DT.T}}_{2^{nd}cycle} + \underbrace{\frac{1}{2}(DT)nT_P}_{3^{rd}cycle}$$

$$= \frac{3}{2}(DT)nT_P + \mathrm{DT}^2 + 2\mathrm{DT}^2$$

$$= \frac{3}{2}(DT)nT_P + \mathrm{DT}^2(1+2)$$

In general,

$$I_{rm} = \frac{\mathrm{u}}{2}(DT)nT_P + \mathrm{DT}^2\big(1 + 2 + \cdots + (\mathrm{u}-1)\big)$$

$$= \frac{\mathrm{u}}{2}(DT)nT_P + \mathrm{DT}^2\left(\frac{\mathrm{u}(\mathrm{u}-1)}{2}\right)$$

$$= \frac{\mathrm{u}}{2}DT^2\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right]$$

Accordingly, the inventory holding cost of the vendor's raw material per unit of time is given by:

$$H_{rm}^{\mathrm{u}} = \frac{h_{rm}}{\mathrm{u}T}\, I_{rm}$$

$$= \frac{h_{rm}}{\mathrm{u}T}\left[\frac{\mathrm{u}}{2}DT^2\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right]\right] = \mathrm{T}.\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right]$$

$$= T.RH_{rm}^{\mathrm{u}} \qquad (11)$$

where RH_{rm}^{u} is the relative raw material inventory holding cost per unit of time for u cycles.

Additionally, since the vendor orders only once every u cycles, the vendor's raw materials ordering cost per unit of time is given by:

$$A_v = \frac{O_v}{\mathrm{u}T} \qquad (12)$$

where O_v is the vendor's ordering cost of raw materials per order.

3.3.3. Multi-To-One (MTO)

In this policy, the vendor orders multiple raw material shipments of equal size (Q_{rm}) during one production cycle in order to satisfy the production in one cycle. Such policy might be intriguing as typically the holding cost is assumed to increase in the downstream direction. Each shipment of size Q_{rm} would be enough to satisfy a fraction of the demand in one cycle. In other words, multiple "m" shipments are required to satisfy the demand in one cycle. Shipments are assumed to be ordered and received when the inventory level reaches zero during the production uptime portion of the cycle. Figure 11 displays the raw material inventory profile under this policy for the same example that was mentioned earlier where 3 raw material shipments are received per cycle (i.e. m = 3).

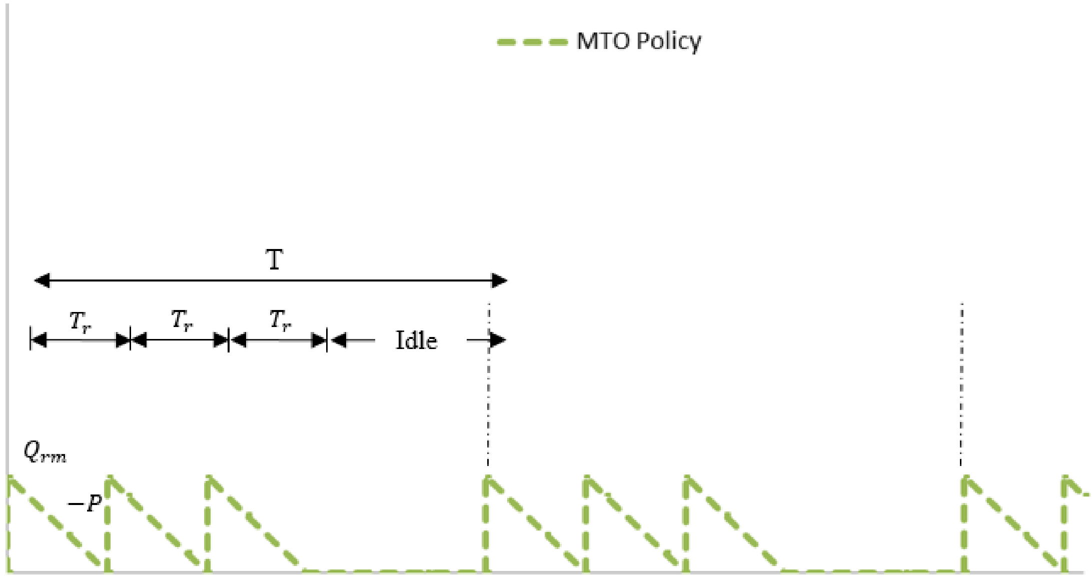

Figure 11: Vendor's raw material inventory profile under MTO policy

The maximum level of inventory is clearly Q_{rm}, which decreases by a constant rate of P during the production uptime. The vendor orders a raw material shipment of size Q_{rm} when the inventory reaches zero, where the time taken to consume this is referred to as the depletion period of raw materials, T_r. This time is given by:

$$T_r = \frac{Q_{rm}}{P} = \frac{DT}{mP} \tag{13}$$

The vendor's inventory is simply the area under the triangles which gives:

$$\frac{1}{2}.T_r.Q_{rm}.\mathrm{m} = \frac{1}{2}\left(\frac{DT}{mP}\right)\left(\frac{1}{m}DT\right)\mathrm{m} = \frac{D^2T^2}{2\mathrm{m}P}$$

The vendor's raw materials inventory holding cost per unit of time is given by:

$$H_{rm}^{m} = \frac{h_{rm}}{T}\left(\frac{D^2T^2}{2\mathrm{m}P}\right) = T.h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) = T.RH_{rm}^{m} \tag{14}$$

where RH_{rm}^{m} is the relative vendor's raw materials inventory holding cost per unit of time.

Since m raw material shipments are received per cycle, the vendor's raw materials ordering cost per unit of time is given by:

$$A_v = \frac{mO_v}{T} \tag{15}$$

where O_v is the vendor's ordering cost of raw materials per order.

Having developed the cost components for all supply chain members under the three raw material replenishment policies, it should be noted that the OTO policy may be considered as a special case of either one of the OTM or MTO policies where $u = 1$ and $m = 1$ in the former and latter policies, respectively.

3.4. Optimization Model

3.4.1. **Three-Stage forward supply chain with CS partnership operating with OTO ordering policy of raw materials (3S-FSC-CS-OTO).** As mentioned earlier, the total long-run average system wide cost includes the following:

- The setup cost at the at the vendor stage (S_v).
- The inventory holding cost of raw materials at the vendor stage (H_{rm}).
- The inventory holding cost of finished products at the vendor stage (H_v).
- The inventory holding cost of finished products at the buyer stage(H_b).
- The buyer's ordering cost of finished products(A_b).
- The vendor's ordering cost of raw materials(A_v).

Using the derived equations of these cost components, the total system cost per unit of time, $K(n, T)$, is mathematically expressed as:

$$K(n,T) = A_v + A_b + S_v + T(RH_{rm} + RH_v + RH_b)$$

$$= \frac{O_v}{T} + \frac{nO_b}{T} + \frac{\mathrm{s}}{T} + T\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$$

$$= \frac{O_v + nO_b + \mathrm{s}}{T} + T\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right) \tag{16}$$

Hence, the optimization model of the three-stage forward supply chain with CS partnership can be stated as follows:

$$\text{Min} \quad K(n,T) = \frac{O_v+nO_b+s}{T} + T\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$$

3.4.2. **Three-Stage forward supply chain with CS partnership operating with OTM ordering policy of raw materials (3S-FSC-CS-OTM).** The total long-run average system wide cost is comprised of the same cost components discussed in the previous sub-section.

Using the derived equations of these cost components, the total system cost per unit of time, $K(n,T,u)$, is mathematically expressed as:

$$K(n,T,u) = A_v + A_b + S_v + T(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)$$

$$= \frac{O_v}{\mathrm{u}T} + \frac{nO_b}{T} + \frac{\mathrm{s}}{T} + T\left(\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$$

$$= \frac{\frac{O_v}{\mathrm{u}}+nO_b+\mathrm{s}}{T} + T\left(\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right) \quad (17)$$

Accordingly, the optimization model of the three-stage forward supply chain with CS partnership can be stated as follows:

$$\text{Min} \quad K(n,T,u) = \frac{\frac{O_v}{\mathrm{u}}+nO_b+\mathrm{s}}{T} + T\left(\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$$

3.4.3. **Three-Stage forward supply chain with CS partnership operating with MTO ordering policy of raw materials (3S-FSC-CS-MTO).** Using the derived equations for the different cost components highlighted in Section 3.4.1, the total system cost per unit of time, $K(n,T,m)$, is mathematically expressed as:

$$K(n,T,m) = A_v + A_b + S_v + T(RH_{rm}^{m} + RH_v + RH_b)$$

$$= \frac{\mathrm{m}O_v}{T} + \frac{nO_b}{T} + \frac{\mathrm{s}}{T} + T\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$$

$$= \frac{mO_v+nO_b+\mathrm{s}}{T} + T\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right) \quad (18)$$

Therefore, the optimization model of the three-stage forward supply chain with CS partnership can be stated as follows:

Min $K(n,T,m) = \frac{mO_v+nO_b+s}{T} + T\left(h_{rm}\left(\frac{D^2}{2mP}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)$

We can easily see that the above optimization models are the general case of the model developed by Braglia and Zavanella [34]. Removing raw material related cost components from the above models results in the special case of a two-stage forward supply chain model provided in [34].

3.5. Solution Algorithm

We can easily see that the objective functions, given by equation 16, 17 and 18 are separable in terms of the cycle time (T). Moreover, for a given number of production batches (n) and fraction of demand or number of cycle covered by one raw material shipment (m or u), all the continuous variables excluding the cycle time can be determined using Equations (5) to (15). To check the convexity of any of the three cost functions having the general form ($K = \frac{\mathrm{A}}{T} + \mathrm{B}T$) with respect to T, we take the second derivative of it which gives:

$$\frac{d^2}{dT^2}(K) = \frac{2\mathrm{A}}{T^3} > 0$$

Since the second derivative is always greater than zero, this gives a proof that the cost function is convex in T, for given values of the other decision variables. Hence, the optimal cycle time (T^*), for fixed values of the other decision variables, can be determined using the first order optimality condition with respect to T .

3.5.1. 3S-FSC-CS-OTO

3.5.1.1. ***Optimal cycle time* (*T^**).** Referring back to Equation (16), we have

$$K(n,T) = \frac{O_v + nO_b + s}{T} + T(RH_{rm} + RH_v + RH_b)$$

Taking the first derivative with respect to T gives

$$\frac{d}{dT}\big(K(n,T)\big) = \frac{-(O_v + nO_b + s)}{T^2} + (RH_{rm} + RH_v + RH_b)$$

Then, the optimal T minimizing the total cost is the one where the first derivative equals to zero. As a result, the optimal T is found to be as follow

$$T^*_{OTO}(n) = \sqrt{\frac{O_v + nO_b + s}{(RH_{rm} + RH_v + RH_b)}}$$

$$= \sqrt{\frac{O_v + nO_b + s}{h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]}} \quad (19)$$

3.5.1.2. ***Optimal number of production batches (n).*** As the number of production batches/runs (n) is a discrete variable, the first order optimality condition is not applicable in this case. Alternatively, the first difference approach will be used to find the optimal n value. This will be done through substituting the expression for the optimal T , given by Equation (19), back in the total cost equation ($K(n,t)$), given by Equation (16), and then using the following two equations to come up with bounds on the optimal n value:

$$K(n^*) - K(n^* - 1) \;\leq 0 \quad (20)$$

$$K(n^* + 1) - K(n^*) \;\geq 0 \quad (21)$$

Substituting Eq. (19) into Eq. (16) gives the following:

$$K(n) = \; 2\sqrt{(RH_{rm} + RH_v + RH_b)(O_v + nO_b + s)} \quad (22)$$

Minimizing $K(n)$ is equivalent to minimizing the terms under the square root. Using Eq. (20) and (22) we have

$$\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + n^*O_b + s)$$
$$- \left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + (n^*-1)O_b$$
$$+ s) \;\leq 0$$

After some algebraic manipulations, this reduces to

$$\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v + \frac{h_b D^2}{2\;P}O_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)s + \frac{h_b D^2}{2\;P}s\right] + \;n^*\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b + \;h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\leq \frac{1}{n^*-1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v + \frac{h_b D^2}{2\;P}O_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)s + \frac{h_b D^2}{2\;P}s\right]$$
$$+ (n^*-1)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b + \;h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$

Separating the decision variable of interest from the other constants results in the following inequality:

$$n^*(n^* - 1) \ \leq \frac{(h_v + h_b)(O_v + s)}{O_b\left[h_{rm} + \ h_b\left(\frac{(P-D)}{D}\right)\right]} \tag{23}$$

In a similar fashion, using Eq. (20) and (21) gives

$$n^*(n^* + 1) \ \geq \frac{(h_v + h_b)(O_v + s)}{O_b\left[h_{rm} + \ h_b\left(\frac{(P-D)}{D}\right)\right]} \tag{24}$$

There is only one integer value for n that satisfies inequalities (23) and (24). To find this optimal value of n, we use Eq. (23) to obtain

$$n^{*^2} - n^* - \beta = 0$$

where $\beta = \frac{(h_v + h_b)(O_v + s)}{O_b\left[h_{rm} + h_b\left(\frac{(P-D)}{D}\right)\right]}$

Solving this quadratic equation results in

$$n^* = \frac{1 \pm \sqrt{1 + 4\beta}}{2}$$

Since $n^* > 0$, then

$$n^* = \frac{1 + \sqrt{1 + 4\beta}}{2} \tag{25}$$

Similarly, from Eq. (24) we have

$$n^* = \frac{-1 + \sqrt{1 + 4\beta}}{2} \tag{26}$$

Accordingly, only one integer will lie between the two values obtained in Eq. (25) and Eq. (26). Hence,

$$n^* = \left\lceil \frac{-1 + \sqrt{1 + 4\beta}}{2} \right\rceil \tag{27}$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

3.5.2. 3S-FSC-CS-OTM

3.5.2.1. ***Optimal cycle time*** $(\boldsymbol{T}^*)$**.** Following the same steps adopted in the previous sub-section to find the optimal T value minimizing $K(n, T, u)$ we obtain

$$T^*_{OTM}(\mathrm{u}, n) = \sqrt{\frac{\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}}{(RH^{\mathrm{u}}_{rm} + RH_v + RH_b)}}$$

$$= \sqrt{\frac{\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}}{\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u} - 1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]}} \quad (28)$$

3.5.2.2. ***Optimal n.*** For a given u, the optimal number of production batches (n^*) is derived as follow:

By substituting Eq. (28) back into Eq. (17), we have:

$$K(n, u) = 2\sqrt{(RH^{\mathrm{u}}_{rm} + RH_v + RH_b)\left(\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}\right)} \quad (29)$$

Minimizing this cost function is equivalent to minimizing the cost components under the square root. Applying the first difference approach for the number of batches (n) using Eq. (20) and Eq. (29) gives

$$n^*(n^* - 1) \leq \frac{(h_v + h_b)\left(\frac{O_v}{\mathrm{u}} + \mathrm{s}\right)}{O_b\left[h_{rm}\left(1 + \frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u} - 1)\right) + h_b\left(\frac{(P-D)}{D}\right)\right]} \quad (30)$$

Similarly, using Eq. (21) and Eq. (29) gives

$$n^*(n^* + 1) \geq \frac{(h_v + h_b)\left(\frac{O_v}{\mathrm{u}} + \mathrm{s}\right)}{O_b\left[h_{rm}\left(1 + \frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u} - 1)\right) + h_b\left(\frac{(P-D)}{D}\right)\right]} \quad (31)$$

Similarly, there is one integer value of n that satisfies inequalities (30) and (31) which is given by:

$$n^* = \left\lceil \frac{-1 + \sqrt{1 + 4\beta}}{2} \right\rceil \tag{32}$$

where

$$\beta = \frac{(h_v + h_b)\left(\frac{O_v}{u} + s\right)}{O_b\left[h_{rm}\left(1 + \frac{P}{D}(u-1)\right) + h_b\left(\frac{(P-D)}{D}\right)\right]}$$

3.5.3. 3S-FSC-CS-MTO

3.5.3.1. ***Optimal cycle time*** $(\boldsymbol{T}^*)$**.** Using the same steps adopted in the previous two subsections, we obtain the optimal T value using the first order optimality condition as follows:

$$T^*_{MTO}(m,n) = \sqrt{\frac{mO_v + nO_b + s}{(RH^m_{rm} + RH_v + RH_b)}}$$

$$= \sqrt{\frac{mO_v + nO_b + s}{h_{rm}\left(\frac{D^2}{2mP}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n}\right) + h_b\left[\frac{D^2}{2\,n\,P} + \left(\frac{(P-D)D}{2P}\right)\right]}} \tag{33}$$

Note that when $m = u = 1$, $T^*_{MTO} = T^*_{OTM} = T^*_{OTO}$.

3.5.3.2. ***Optimal n.*** For a given m, the optimal number of production batches (n^*) is derived as follow:

By substituting Eq. (35) in Eq. (18), we have:

$$K(n,m) = 2\sqrt{(RH^m_{rm} + RH_v + RH_b)(mO_v + nO_b + s)} \tag{34}$$

Similarly, it suffices to minimize the terms under the square root in order to minimize the cost function $K(n,m)$. Using Eq. (20) and Eq. (34) gives

$$n^*(n^* - 1) \le \frac{(h_v + h_b)(mO_v + s)}{O_b\left[\frac{h_{rm}}{m} + h_b\left(\frac{(P-D)}{D}\right)\right]} \tag{35}$$

Similarly, using Eq. (21) and Eq. (34) gives

$$n^*(n^*+1) \geq \frac{(h_v+h_b)(mO_v+s)}{O_b\left[\frac{h_{rm}}{m}+ h_b\left(\frac{(P-D)}{D}\right)\right]} \tag{36}$$

Using Eq. (35) and Eq. (36), we have one integer value of n satisfying both inequalities which is given by:

$$n^* = \left\lceil \frac{-1+\sqrt{1+4\beta}}{2} \right\rceil \tag{37}$$

where

$$\beta = \frac{(h_v+h_b)(mO_v+s)}{O_b\left[\frac{h_{rm}}{m}+ h_b\left(\frac{(P-D)}{D}\right)\right]}$$

We can clearly see that when $u = m = 1$, Eq. (32) and Eq. (37) reduce to Eq. (27). As pointed out, this justifies that fact that 3S-FSC-CS-OTO model is a special case of the other two models, namely 3S-FSC-CS-OTM and 3S-FSC-CS-MTO. Hence, dealing with those two models will guarantee the optimality of the decision made without the need to use 3S-FSC-CS-OTO model. In the following sections, models FSC-CS-OTM and 3S-FSC-CS-MTO will be used.

In the previous analysis, a closed form solution has been derived for finding the optimal T and n values for given u and m. However, it is difficult to derive a similar closed form expression for the optimal u and m values. Nevertheless, Eq. (17) and Eq. (18) clearly show that the cost function is convex in u and m, respectively. As such, in order to find the optimal solution for the integer variables u and m, and hence the optimal cost function, the following algorithm is developed:

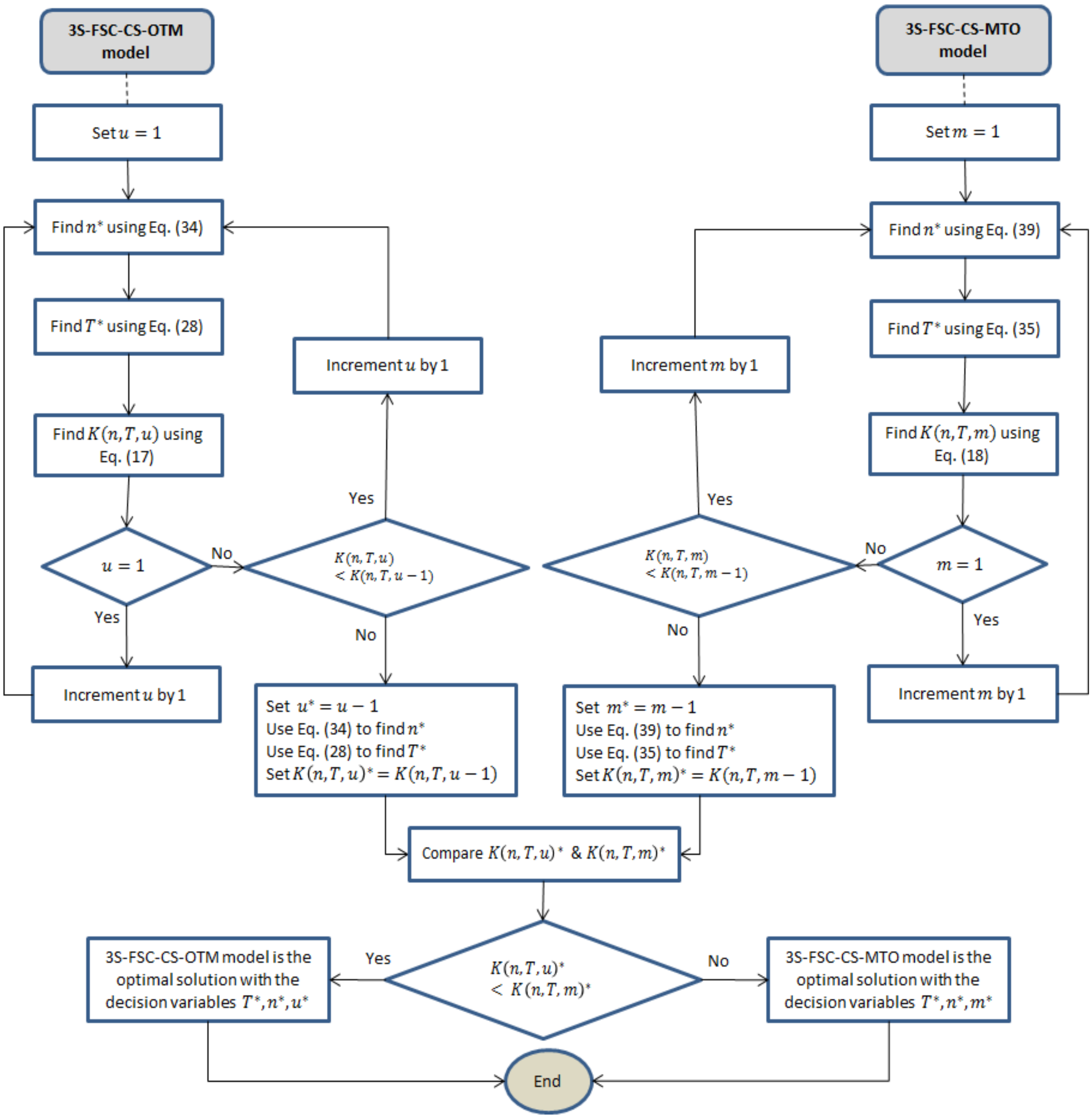

Figure 12: 3S-FSC solution algorithm

The algorithm in Figure 12 seeks to find the optimal solution through following the below steps:

- **Step 1:** Using 3S-FSC-CS-OTM model, set $u = 1$ and Find the optimal number of batches (n^*) using Eq. (34). After that, find the optimal cycle time (T^*) and the total cost $K(n,T,u)$ using Eq. (28) and (17), respectively.
- **Step 2:** Increment u by 1 and repeat Step 1. Keep doing this step unless total cost increases by increasing u.
- **Step 3:** Find the optimal u, n and T that give the lowest total cost $K(n,T,u)$.
- **Step 4:** Repeat the steps from 1 to 3 using 3S-FSC-CS-MTO model to find the optimal m, n and T that give the lowest total cost $K(n,T,m)$.

- **Step 5:** Compare the two cost functions, $K(n, T, u)$ and $K(n, T, m)$, and determine the more cost efficient solution. This will be the optimal solution.

3.6. Computational Experiments

In this section, the proposed solution algorithm is first applied towards solving a numerical example to illustrate the impact of raw materials ordering policies on the total system cost. This is followed by a sensitivity analysis which is performed to provide more insights on the impact of key problem parameters on the behavior of the developed models.

3.6.1. **Numerical example (base case).** The input parameters to this example (base case) are: $\mathrm{S} = 200, O_b = 100, O_v = 200, \mathrm{D} = 2000, \mathrm{P} = 4000, h_v = 3, h_b = 4$ and $h_{rm} = 2$. To check the robustness of the algorithm, two opposite scenarios are used: the first scenario deals with the case where the ordering cost of raw materials is very low (0.01 of the base case value: $O_v = 2$ and the second scenario is where the ordering cost of raw materials is quite high (30 times the base case value: O_v=6000) while fixing all the other parameters at their base value. The results of the algorithm are shown in Figure 13.

As anticipated, when the ordering cost of raw material is low, 3S-FSC-CS-MTO model is giving the optimal solution with a total cost of 1619 with $m = 7$ compared with 3S-FSC-CS-OTM model which gives the lowest total cost of 1822 when $u = 1$. However, when the ordering cost of raw material is high, 3S-FSC-CS-OTM model is giving the optimal solution with a total cost of 8089 with $u = 4$ as opposed to 3S-FSC-CS-MTO model which gives the lowest total cost of 8890 when $m = 1$.

This result validates the following: when the ordering cost of raw material is high compared to the other cost components (e.g. holding cost), it is more cost effective to order one large raw material shipment that covers multiple production cycles, and vice versa; it would be better to have multiple shipments of raw material per cycle when the ordering cost of raw material is low compared to the other cost components. Moreover, we can see that both models give the same solution when $m = u = 1$ which is the special case that was mentioned earlier (3S-FSC-CS-OTO model). Tables 5 and 6 show detailed results of the 3S-FSC algorithm in finding the optimal solution along with the rest of the decision variables.

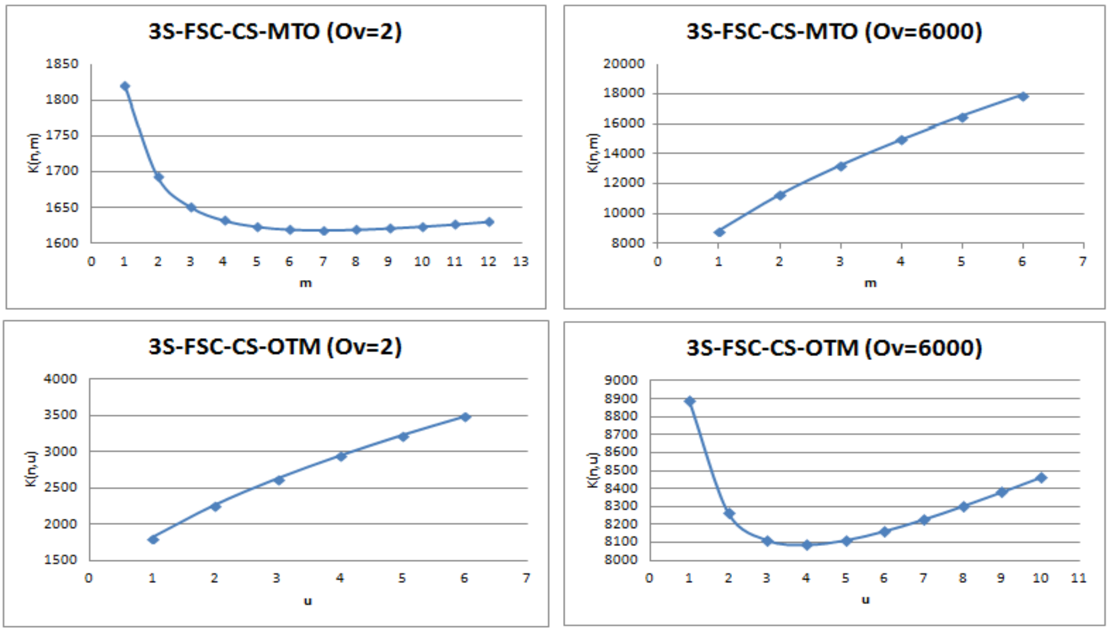

Figure 13: FSC algorithm results

Referring to Tables 5Table 5 and 6, when $O_v = 2$ the optimal solution is to have 7 orders of raw material per cycle ($m = 7$) where also $n = 8,\ T = 0.313745\ and\ K = 1619.149$. However, when $O_v = 6000$ the optimal solution is to have one raw material shipment to cover 4 cycles ($u = 4$) with $n = 12,\ T = 0.435112\ and\ K = 8087.851$.

Table 5: 3S-FSC algorithm results (O_v =2)

$O_v = 2$				
u	m	n^*	T^*	$K(n)$
1	-	7	0.26022	1821.538
2	-	5	0.199121	2269.978
3	-	5	0.171194	2636.386
4	-	4	0.149429	2951.229
5	-	4	0.136235	3235.583
6	-	3	0.123288	3493.168
-	1	7	0.26022	1821.538
-	2	8	0.288208	1693.222
-	3	8	0.297963	1651.212
-	4	8	0.303775	1632.789
-	5	8	0.307875	1624.038
-	6	8	0.311076	1620.185
-	**7**	**8**	**0.313745**	**1619.149**
-	8	8	0.316073	1619.877
-	9	9	0.324345	1621.727
-	10	9	0.326302	1624.261

Table 6: 3S-FSC algorithm results (O_v =6000)

$O_v = 6000$				
u	m	n^*	T^*	$K(n)$
1	-	38	1.437551	8890.119
2	-	21	0.799798	8264.583
3	-	15	0.560817	8113.158
4	**-**	**12**	**0.435221**	**8087.851**
5	-	9	0.356199	8113.432
6	-	8	0.303775	8163.945
7	-	7	0.265445	8228.782
8	-	6	0.236082	8302.209
9	-	6	0.21398	8380.864
10	-	5	0.194983	8462.269
-	1	38	1.437551	8890.119
-	2	58	2.208676	11309.94
-	3	74	2.792929	13297.87
-	4	87	3.27972	15022.62
-	5	98	3.705018	16566.72
-	6	108	4.087424	17977.09

Furthermore, to validate the developed solution algorithm, the models have been coded and solved using Lingo. Referring to Lingo solution, we found that that the algorithm is giving exactly the same optimal solution which validates the effectiveness of the 3S-FSC algorithm.

3.6.2. **Sensitivity analysis.** This subsection presents extensive computational experiments for different problem instances under different settings of the problem parameters. In particular, we seek to study the impact of changing the cost parameters (holding, setup and ordering costs) on the behavior of the model. The results of this analysis shall illustrate the relevance and importance of the derived models and provide more insights on the relationship between problem parameters and its impact on the decision variables. Then, a comparison between the derived algorithm and the two-stage model of Braglia and Zavanella [34] is carried out in order to analyze the impact of adding one more stage, raw materials procurement, to the centralized supply chain system.

3.6.2.1. ***Impact of buyer's ordering and holding costs.*** In order to assess the impact of the buyer's ordering and holding costs on the optimal solutions obtained, those cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. As the ordering cost typically increases in the upstream direction, five different values for O_b are chosen through multiplying its base value of $O_b = 100$ by 0.01, 0.10, 1, 1.5 and 2. Those multipliers are chosen to cover a wide range and at the same time to be consistent with the ordering cost of the vendor ($O_v \geq O_b$). Similarly, as the holding cost typically increases in the downstream direction, five different values for h_b are chosen for each of the above mentioned O_b values; $h_b = 3\,(= h_v), 3.5, 4, 4.5\ and\ 5$. Hence, a total of 25 problem instances are solved under the two different ordering policies, and the results are listed in Table 7.

Table 7: Effect of changing buyer's ordering and holding costs

		3S-FSC-CS-OTM				3S-FSC-CS-MTO			
O_b	h_b	n	u	T	TC	n	m	T	TC
1	3	22	1	0.4001	2,109.55	22	1	0.4001	2,109.55
1	3.5	22	1	0.3816	2,211.64	22	1	0.3816	2,211.64
1	4	22	1	0.3655	2,309.23	22	1	0.3655	2,309.23
1	4.5	21	1	0.3504	2,402.86	21	1	0.3504	2,402.86
1	5	21	1	0.3378	2,492.94	21	1	0.3378	2,492.94
10	3	7	1	0.4006	2,346.43	7	1	0.4006	2,346.43
10	3.5	7	1	0.3824	2,458.22	7	1	0.3824	2,458.22
10	4	7	1	0.3665	2,565.15	7	1	0.3665	2,565.15
10	4.5	7	1	0.3524	2,667.80	7	1	0.3524	2,667.80
10	5	7	1	0.3398	2,766.64	7	1	0.3398	2,766.64
100	3	2	1	0.3873	3,098.39	2	1	0.3873	3,098.39
100	3.5	2	1	0.3703	3,240.37	2	1	0.3703	3,240.37
100	4	2	1	0.3554	3,376.39	2	1	0.3554	3,376.39
100	4.5	2	1	0.3422	3,507.14	2	1	0.3422	3,507.14
100	5	2	1	0.3303	3,633.18	2	1	0.3303	3,633.18
150	3	2	1	0.4183	3,346.64	2	1	0.4183	3,346.64
150	3.5	2	1	0.4000	3,500.00	2	1	0.4000	3,500.00
150	4	2	1	0.3839	3,646.92	2	1	0.3839	3,646.92
150	4.5	2	1	0.3696	3,788.14	2	1	0.3696	3,788.14
150	5	2	1	0.3568	3,924.28	2	1	0.3568	3,924.28
200	3	2	1	0.4472	3,577.71	2	1	0.4472	3,577.71
200	3.5	2	1	0.4276	3,741.66	2	1	0.4276	3,741.66
200	4	2	1	0.4104	3,898.72	2	1	0.4104	3,898.72
200	4.5	2	1	0.3951	4,049.69	2	1	0.3951	4,049.69
200	5	2	1	0.3814	4,195.24	2	1	0.3814	4,195.24

It is clearly seen that, for both models, as the value of the buyer's ordering cost increases, it is more economical to have fewer orders of larger sizes which is reflected in the decreasing the value of n while T stays almost the same. However, altering the buyer's ordering cost has no effect on the raw materials decision variables (u and m).

In addition, as the value of the buyer's holding cost increases, the buyer is clearly better off making more frequent orders of smaller sizes (as can be clearly seen for instance when $O_b = 1$). This reduction in the ordered lot size is achieved through decreasing the cycle time, T, as clearly seen in the table where T gets smaller as h_b is getting higher. Moreover, increasing the value of the buyer's holding cost has no impact on the order frequency (i.e. number of batches) for $O_b > 1$ but it does reduce the batch size through reducing the cycle time (T).

In addition, we have also experimented with higher values for the buyer's holding cost; namely $h_b = 15, 30$, 150 and 300, and we observed that increasing the value of the buyer's holding cost results in less frequent orders of raw materials (u gets higher in 3S-FSC-CS-OTM model while m stays at minimum of 1 order in 3S-FSC-CS-MTO model) in an attempt to reduce the ordering cost of raw materials by covering more than one cycle with one raw material shipment. This is happening due to the smaller values of the cycle time (T) as holding cost gets higher. Hence, the effect of varying the buyer's ordering and holding costs may extend beyond the finished product ordering policy, in terms of frequency and batch size, to include the raw materials ordering decision at the vendor's facility as well.

We can see that 3S-FSC-CS-OTM model is always having similar total cost compared to 3S-FSC-CS-MTO for the chosen values of O_b and h_b in Table 7. However, for the higher experimented values of h_b, 3S-FSC-CS-OTM model is always having lower total cost compared to 3S-FSC-CS-MTO. This better performance is achieved from taking the advantage of reducing the ordering cost of raw materials where the demand in more than one cycle is covered from one raw material shipment as the value of T gets smaller where 3S-FSC-CS-MTO model will order the minimum amount of one order per cycle. Both models give exactly the same optimal solution only when the decision variables u and m are the same (=1). Besides, higher values of O_b (> 200) have been experimented and results showed that both models give exactly the same optimal solution for all values of $h_b$and that the decision variables $n, m\ and\ u$ are always at their minimal values (=1).

3.6.2.2. ***Impact of vendor's holding cost and manufacturing setup cost.*** To gain better insights on the impact of the vendor's finished product holding cost and the manufacturing setup cost on the optimal solutions obtained, a two-way sensitivity analysis is performed in which those cost parameters are altered while keeping all other problem parameters fixed as in the base case. To cover a wide range, five different values for S are chosen through multiplying its base value of $S = 200$ by 0.01, 0.10, 1, 10 and 15. For the finished product holding cost, as this cost typically increases in the downstream direction ($h_{rm} \leq h_v \leq h_b$), five different values for h_v are chosen for each of the aforementioned S values; $h_v = 2 (= h_{rm}), 2.5, 3, 3.5$ *and* $4 (= h_b)$. This results in a total of 25 problem instances with the solutions for each instance under both raw material ordering policies, summarized in Table 8.

The holding cost of the vendor is slightly affecting the decision variables n and T but has no effect on the raw materials related decision variables m and u. As can be seen from the above table, for high values of S, increasing the finished product holding cost at the vendor's stage results in a higher number of batches (n) to be shipped to the buyer and smaller cycle time. This will slightly reduce the vendor's inventory and hence its holding cost. However, the other decision variables, namely m and u, remain the same for all values of h_v when all other cost parameters are being fixed.

We can clearly see that the manufacturing setup cost has an impact on the decision variables n, T, m and u. Since the setup cost takes place only once at the beginning of the cycle, increasing the setup cost results in increasing the cycle time to overcome the high setup cost. Larger cycle time will result in lower setup cost per cycle, or lower setup cost per unit time. Moreover, it is noticed that increasing the setup cost increases the number of batches shipped to the buyer (n) but the quantity shipped remains almost the same so that vendor's and buyer's holding costs are minimized.

Looking again at Table 8, we can clearly see that the two models (3S-FSC-CS-OTM and 3S-FSC-CS-MTO) act oppositely in response to increasing values of the setup cost (S). Model 3S-FSC-CS-OTM reduces the number of cycles covered in one raw material shipment (u) with increasing S in an attempt to minimize the inventory holding cost of raw materials. On the other hand, model 3S-FSC-CS-MTO increases the number of raw material shipments per cycle (m) with increasing S also in an attempt to minimize the inventory holding cost of raw materials. This will result in lower holding cost but higher ordering cost. However, a tradeoff between ordering and holding costs will lead to the minimum cost.

Table 8: Effect of vendor's holding cost and manufacturing setup cost on the behavior of the models

		3S-FSC-CS-OTM				3S-FSC-CS-MTO			
S	h_v	n	u	T	TC	n	m	T	TC
2	2	1	2	0.1589	2,542.44	2	1	0.2989	2,689.98
2	2.5	1	2	0.1565	2,581.86	2	1	0.2948	2,727.09
2	3	1	2	0.1542	2,620.69	2	1	0.2909	2,763.69
2	3.5	1	2	0.1519	2,658.95	2	1	0.2872	2,799.82
2	4	1	2	0.1498	2,696.67	2	1	0.2835	2,835.49
20	2	1	2	0.1658	2,653.30	2	1	0.3055	2,749.55
20	2.5	1	2	0.1633	2,694.44	2	1	0.3013	2,787.47
20	3	1	2	0.1609	2,734.96	2	1	0.2974	2,824.89
20	3.5	1	2	0.1586	2,774.89	2	1	0.2935	2,861.82
20	4	1	2	0.1563	2,814.25	2	1	0.2898	2,898.28
200	2	2	1	0.3651	3,286.34	2	1	0.3651	3,286.34
200	2.5	2	1	0.3602	3,331.67	2	1	0.3602	3,331.67
200	3	2	1	0.3554	3,376.39	2	1	0.3554	3,376.39
200	3.5	2	1	0.3508	3,420.53	2	1	0.3508	3,420.53
200	4	2	1	0.3464	3,464.10	2	1	0.3464	3,464.10
2000	2	5	1	0.8660	6,235.38	5	2	0.9672	5,996.67
2000	2.5	5	1	0.8601	6,278.54	6	2	0.9931	6,041.52
2000	3	5	1	0.8542	6,321.39	6	2	0.9864	6,082.76
2000	3.5	5	1	0.8485	6,363.96	6	2	0.9798	6,123.72
2000	4	5	1	0.8429	6,406.25	6	2	0.9733	6,164.41
2500	2	5	1	0.9428	6,788.23	6	3	1.1428	6,475.60
2500	2.5	5	1	0.9363	6,835.20	7	3	1.1655	6,521.03
2500	3	6	1	0.9597	6,877.50	7	3	1.1581	6,562.52
2500	3.5	6	1	0.9541	6,917.37	7	3	1.1509	6,603.75
2500	4	6	1	0.9487	6,957.01	7	3	1.1438	6,644.73

Because of the above mentioned counteract behaviors with S, it is obviously seen that 3S-FSC-CS-OTM model gives lower cost solution for low values of S while 3S-FSC-CS-MTO model gives better solution for higher values of S (above and below 200 in this example). This difference in the total cost between the two models gets larger as S goes lower/higher than the midpoint value. Either way, model 3S-FSC-CS-MTO always results in a larger cycle time (T) and larger/equal number of batches (n).

3.6.2.3. ***Impact of vendor's ordering and holding costs of raw materials.*** We also analyze the impact of the vendor's ordering and holding costs of raw materials on the decision variables and hence optimal cost obtained. Those cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. As the ordering cost typically increases in the upstream direction, five different values for O_v are chosen: $O_v = O_b$, 200, 400, 800 and 1000. On the other hand, since the holding cost typically increases in the downstream direction ($h_{rm} \leq h_v$), five different values for h_{rm} are chosen for each of the aforementioned O_v values; $h_{rm} = 0.5, 1, 1.5, 2.5\ and\ 3(= h_v)$. This results in a total of 25 problem instances where the solutions for each instance under the two different raw material ordering policies are summarized in Table 9.

It is evident that changing the cost parameters related to raw materials has a major impact on the decision variables of the models. Increasing the ordering cost of raw materials slightly impacts the cycle time (T) when operating under 3S-FSC-CS-OTM model. However, this has a tangible impact on the cycle time while operating under 3S-FSC-CS-MTO model. Increasing raw materials ordering cost results in increasing the cycle time. As we can see from Table 9, 3S-FSC-CS-MTO model has higher or equal cycle time for all the solutions, and the difference is getting higher for larger values of raw materials ordering cost.

It is also noted that model 3S-FSC-CS-OTM slightly increases the number of batches shipped to the buyer (n) as the cycle time is being slightly affected by increasing O_v. However, this has a higher impact on the number of batches while operating under 3S-FSC-CS-MTO model because increasing O_v largely affects (increases) T in this model which results in a high increase in the number of batches (n).

As expected, it is clearly seen that higher values of O_v increase the number of production cycles covered in one raw material shipment (u) when operating under 3S-FSC-CS-OTM model. This is obviously happening to reduce the ordering cost of raw materials per cycle which is the main advantage behind using such an ordering policy of raw materials. However, when operating under 3S-FSC-CS-MTO model the higher values of O_v will minimize the number of raw materials orders per cycle to its minimum (i.e. m=1), as seen in Table 9, and increase T as much as possible to reduce the ordering cost of raw materials per cycle ($\frac{m\,O_v}{T}$).

Table 9: Effect of vendor's ordering and holding costs of raw materials on the behavior of the models

		3S-FSC-CS-OTM				3S-FSC-CS-MTO			
O_v	h_{rm}	n	u	T	TC	n	m	T	TC
100	0.5	2	1	0.3536	2,828.43	2	1	0.3536	2,828.43
100	1	2	1	0.3430	2,915.48	2	1	0.3430	2,915.48
100	2	2	1	0.3244	3,082.21	2	1	0.3244	3,082.21
100	2.5	2	1	0.3162	3,162.28	2	1	0.3162	3,162.28
100	3	2	1	0.3086	3,240.37	2	1	0.3086	3,240.37
200	0.5	2	2	0.3333	3,000.00	3	1	0.4526	3,093.00
200	1	2	1	0.3757	3,193.74	2	1	0.3757	3,193.74
200	2	2	1	0.3554	3,376.39	2	1	0.3554	3,376.39
200	2.5	2	1	0.3464	3,464.10	2	1	0.3464	3,464.10
200	3	2	1	0.3381	3,549.65	2	1	0.3381	3,549.65
400	0.5	2	3	0.3266	3,265.99	3	1	0.5132	3,507.14
400	1	2	2	0.3381	3,549.65	3	1	0.4954	3,633.18
400	2	3	1	0.4648	3,872.98	3	1	0.4648	3,872.98
400	2.5	3	1	0.4514	3,987.48	3	1	0.4514	3,987.48
400	3	2	1	0.3904	4,098.78	2	1	0.3904	4,098.78
800	0.5	2	4	0.3303	3,633.18	4	1	0.6693	4,183.30
800	1	2	3	0.3266	4,082.48	4	1	0.6441	4,347.41
800	2	2	2	0.3443	4,647.58	3	1	0.5586	4,654.75
800	2.5	3	1	0.5425	4,792.36	3	1	0.5425	4,792.36
800	3	3	1	0.5278	4,926.12	3	1	0.5278	4,926.12
1000	0.5	2	4	0.3438	3,781.53	4	1	0.7155	4,472.14
1000	1	2	3	0.3425	4,281.74	4	1	0.6885	4,647.58
1000	2	2	2	0.3651	4,929.50	4	1	0.6426	4,979.96
1000	2.5	4	1	0.6228	5,138.09	4	1	0.6228	5,138.09
1000	3	4	1	0.6047	5,291.50	4	1	0.6047	5,291.50

Turning to the raw materials holding cost, when operating under 3S-FSC-CS-OTM model, increasing this holding cost results in increasing the cycle time (T) when $O_v >$ 800. Some solutions result in a smaller T when increasing h_{rm} for $O_v < 800$. This only happens when increasing h_{rm} does not change the decision variable u to compensate for the increase in the holding cost. On the other hand, when operating under 3S-FSC-CS-MTO model, increasing the holding cost of raw materials results in decreasing the cycle time (T) for all the instances to minimize the inventory level.

It is also clearly seen that when h_{rm} is getting higher, the number of batches shipped to the buyer (n) is getting higher when operating under 3S-FSC-CS-OTM model and lower when operating under 3S-FSC-CS-MTO model. This is clearly seen when h_{rm} is relatively low compared to O_v ($O_v \geq 400$ for 3S-FSC-CS-OTM model and $O_v \geq 200$ for 3S-FSC-CS-MTO model). In addition, we can see that 3S-FSC-CS-OTM model reduces the number of production cycles covered by one raw material shipment (u) as h_{rm} is getting higher. This is obviously happening to minimize the inventory levels as its holding cost is getting higher and reaches its minimum value, i.e. $u = 1$, when the holding cost is relatively high compared to the ordering cost. However, altering h_{rm} value seems to have no noticeable impact on the decision variable m when operating under 3S-FSC-CS-MTO for the chosen values of cost parameters. This is happening due the fact that the ordering cost is relatively high compared to the holding cost. In this case, we can see that 3S-FSC-CS-OTM model is always performing similar or better than 3S-FSC-CS-MTO model. Obviously, 3S-FSC-CS-OTM model gives exactly the same solution as 3S-FSC-CS-MTO model when its decision variable u is at minimum, i.e. $u = m = 1$.

To get better insights on the two models and to see the changes on the decision variable m compared to u, we have to go for lower values of O_v. To that sake, the base value of the buyer's ordering costs ($O_b = 100$) has to be reduced first to be aligned with O_v ($O_v \geq O_b$). Let $O_b = 10$, then three different values for O_v are chosen: $O_v = O_b$, 25 and 50 while all other cost parameters are kept the same. This results in a total of 15 problem instances for lower raw materials ordering cost with the solutions for each instance under the two different ordering policies models, being summarized in Table 10.

Now when the ordering cost of raw materials is relatively low compared to the holding cost, we can see that 3S-FSC-CS-MTO model outperforms 3S-FSC-CS-OTM model through taking advantage of the multi orders per cycle which reduces the holding cost of raw materials. Increasing the ordering cost of raw materials slightly increases the cycle time (T) when operating under 3S-FSC-CS-OTM model. However, when operating under 3S-FSC-CS-MTO model, the cycle time got smaller when m does not change with increasing h_{rm} and got higher when m changes with increasing h_{rm}.

However, model 3S-FSC-CS-MTO always results in higher or equal cycle time for all the instances tested.

In addition, it is clearly seen that lower values of O_v increase the number of raw material shipments per cycle (m) when operating under 3S-FSC-CS-MTO model. This is obviously happening to reduce the holding cost of raw materials per cycle which is an advantage of using such an ordering policy of raw materials. However, when operating under 3S-FSC-CS-OTM model the low values of O_v relative to h_v will minimize the number of cycles to cover by one raw material shipment to its minimum (i.e. $u = 1$), as seen in Table 10, which was not the case for high values of O_v as in Table 9 ($u > 1$ and $m = 1$).

Table 10: Effect of vendor's ordering and holding costs of raw materials on the behavior of the models for lower raw materials ordering cost

		3S-FSC-CS-OTM				3S-FSC-CS-MTO			
O_v	h_{rm}	n	u	T	TC	n	m	T	TC
10	0.5	6	1	0.3087	1,749.29	6	2	0.3215	1,741.65
10	1	5	1	0.2850	1,824.28	6	2	0.3144	1,781.39
10	2	5	1	0.2651	1,961.63	6	3	0.3153	1,839.38
10	2.5	5	1	0.2566	2,026.82	6	4	0.3219	1,864.14
10	3	5	1	0.2488	2,089.98	6	4	0.3184	1,884.14
25	0.5	6	1	0.3172	1,797.22	6	1	0.3172	1,797.22
25	1	6	1	0.3040	1,874.83	6	2	0.3308	1,874.39
25	2	5	1	0.2726	2,017.42	6	2	0.3171	1,955.34
25	2.5	5	1	0.2639	2,084.47	6	2	0.3108	1,994.58
25	3	5	1	0.2559	2,149.42	6	3	0.3296	2,032.65
50	0.5	6	1	0.3308	1,874.39	6	1	0.3308	1,874.39
50	1	6	1	0.3171	1,955.34	6	1	0.3171	1,955.34
50	**2**	**5**	**1**	**0.2847**	**2,107.13**	**6**	**2**	**0.3417**	**2,107.13**
50	2.5	5	1	0.2756	2,177.15	6	2	0.3350	2,149.42
50	3	5	1	0.2673	2,244.99	6	2	0.3286	2,190.89

In this case, we can see that 3S-FSC-CS-MTO model is always performing similar or better than 3S-FSC-CS-OTM model. Again, 3S-FSC-CS-MTO model gives exactly the same solution as 3S-FSC-CS-OTM model when its decision variable m is at minimum, i.e. $m = u = 1$. However, one of the instances gave exactly the same total cost but with different values of the decision variable (bolded in Table 10).

Combining the results from Tables 9 and 10, we can see that when O_v is relatively high compared to h_{rm} it is more economically and cost effective to operate under 3S-FSC-CS-OTM model to reduce the impact of high raw material ordering cost through having fewer orders of larger sizes of raw materials such that one raw material shipment suffices the production during multiple cycles. On the other hand, when O_v is relatively low compared to h_{rm} it is more cost effective to operate under 3S-FSC-CS-MTO model to reduce the impact of high raw material inventory holding cost via having more orders of smaller sizes of raw materials within each production cycle.

3.6.2.4. ***Impact of adding raw materials stage in the centralized decision.*** To study the impact of including raw materials related costs on the centralized supply chain system under consideration, we have adopted the same example used in the two-stage model of Braglia and Zavanella [34]. In their model, the input parameters are $S = 400$, $O_b = 25$, $D = 1000$, $P = 3200$, $h_v = 4$, $h_b = 5$. The optimal cost resulted using their model was 2,035.9 with $n = 4$ and $T = 0.4914$.

Before analyzing the impact of adding raw materials related costs, we set the vendor's raw materials ordering and holding costs in our model to zero to make sure that our model reduces to Braglia and Zavanella [34] model. The results showed that the solutions were exactly the same which validates that their model is a special case of our model.

In addition, we have also experimented the effect of adding one more stage to the system through carrying out a two-way sensitivity analysis for different values of O_v and h_{rm}, and then comparing the results with the model in [34]. The results showed that adding raw material stage to the centralized decision has an impact on the batch size (Q), where, for instance, the batch size in the 3S-FSC-CS model ranged between 97% and 123% of the batch size in Braglia and Zavanella [34] model when simultaneously changing O_v from 25 to 300 and h_{rm} from 0.5 to 4.

In addition to that, an impact is seen as well on the frequency where the cycle time ranged between 0.4422 (90% of B & Z model frequency) and 0.5794 (118% of B & Z model frequency) for the three-stage supply chain compared to 0.4914 in Braglia and Zavanella [34] example. This flexibility and difference in the decision variables play a crucial rule in optimizing the whole chain and providing a better cost efficient solution.

Hence, including a new stage to the centralized decision does have a major impact on the related decision variables and hence on the long-run system wide cost.

3.7. Conclusion

To conclude with, this chapter illustrated the impact of adding raw material stage in optimizing a centralized supply chain comprised of a single vendor/manufacturer and a single buyer operating under CS partnership. Three mathematical models have been developed according to the three raw material replenishment policies used.

Mathematical models and results showed that 3S-FSC-CS-OTO model is a special case of the other two models. Besides, we have seen that the model developed by Braglia and Zavanella [34] is a special case of our developed model and could be achieved by removing raw material cost components. In addition to that, results validated that raw material cost components do have a major impact on the related decision variables and hence on the long-run system wide cost.

To expand the work done in this chapter, new models will be developed to study the effect of remanufacturing and production sequence on the chain wide cost. The following chapter will illustrate a three-stage closed loop supply chain system that operates under consignment stock partnership (3S-CLSC-CS).

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Chapter 4: Three-Stage Closed Loop SC with CS Partnership (3S-CLSC-CS)

The review of the literature provided in Chapter 2 reveals that the vast majority of the work addressing closed loop supply chain assumes the production sequence to be a given input to the optimization problem, rather than a decision variable to be optimized. The most explored production sequences are:

- Sequence (1, M) where a remanufacturing batch is produced first followed by a setup and then *M* newly manufactured batches.
- Sequence (R, 1) where *R* remanufacturing batches are produced first, followed by a setup and then a single newly manufactured batch.
- The generalized sequence of (R, M) where *R* consecutive remanufacturing batches are produced first followed by a setup and then *M* consecutive newly manufactured batches.

It can be noted in all of the above sequences that remanufacturing of returned items takes place before the production of new items. This may be attributed partially to the fact that a two-stage supply chain model, the most commonly treated supply chain configuration in the literature, does not account for the cost of incoming raw material at the vendor stage, but it does typically account for the cost of holding returned products before being remanufactured. Hence, to reduce the holding cost of those returned products, it makes sense to start the production cycle with the remanufacturing of those products.

In this chapter, however, we study the impact of incorporating raw material related costs (ordering and holding) on the production sequence and the chain wide total cost. To that end, a centralized supply chain composed of a single supplier, a single vendor/manufacturer and a single buyer operating with CS partnership is analyzed, where it is now assumed that a fraction of the demand can be recovered and remanufactured. In this case, the end customer demand may be fulfilled either from producing newly manufactured products using newly delivered raw materials (product type 1) or remanufactured products using the returned products (product type 2) at each production run [24], where each of the raw materials ordering policies that were mentioned earlier are explored. Unlike most of the existing work, the vendor is producing type 1 and type 2 products with no fixed sequence of production, and hence the production sequence is a decision variable that is optimized through the mathematical model. Similar to the work of Hariga et al [24], it is assumed that the quality of product type 2 (remanufactured product) is as good as that of product type 1 (newly manufactured product). Under CS partnership, the manufacturer will ship the manufactured and remanufactured batches as soon as they are produced in order to keep the inventory of finished products at a minimum level and accordingly minimize its holding cost of finished products. Hence, the manufacturer will keep on shipping the batches to the buyer as long as production is running where the last delivery to the buyer is made as soon as the production ceases.

4.1. Models Development

In order to develop the mathematical models for the three-stage closed loop supply chain under the three raw material replenishment polices, we adopt the same notations as in Hariga et al. [24] throughout this chapter. Let:

Q_i: production batch size for product type i, $where\ i = 1{,}2$

n_i: number of production batches for product type i, $where\ i = 1{,}2$ in one cycle

n: $total$ number of production batches $= n_1 + n_2$

r: $return\ rate$

d_1: $demand\ rate\ for\ product\ type\ 1,\ where\ d_1 = D - r$

d_2: $demand\ rate\ for\ product\ type\ 2,\ where\ d_2 = r$

P_1: $manufacturing\ rate$

R: $remanufacturing\ rate$

During the production uptime portion of the cycle, the vendor ships equal batches of size Q_i (Q_1 for product type 1 and Q_2 for product type 2), and the time taken to produce one batch of size Q_i is called the production time for product type i (T_{P_i}). The batch size and the production time for one batch of product type i can be determined as a function of cycle time (T) and number of production batches for each product (n_i) as follows:

$$Q_i = \frac{d_i T}{n_i} = q_i T \tag{38}$$

$$T_{P_i} = \frac{Q_i}{P_i} = \frac{\frac{d_i T}{n_i}}{P_i} = \frac{d_i}{P_i n_i} T = t_{P_i} T \tag{39}$$

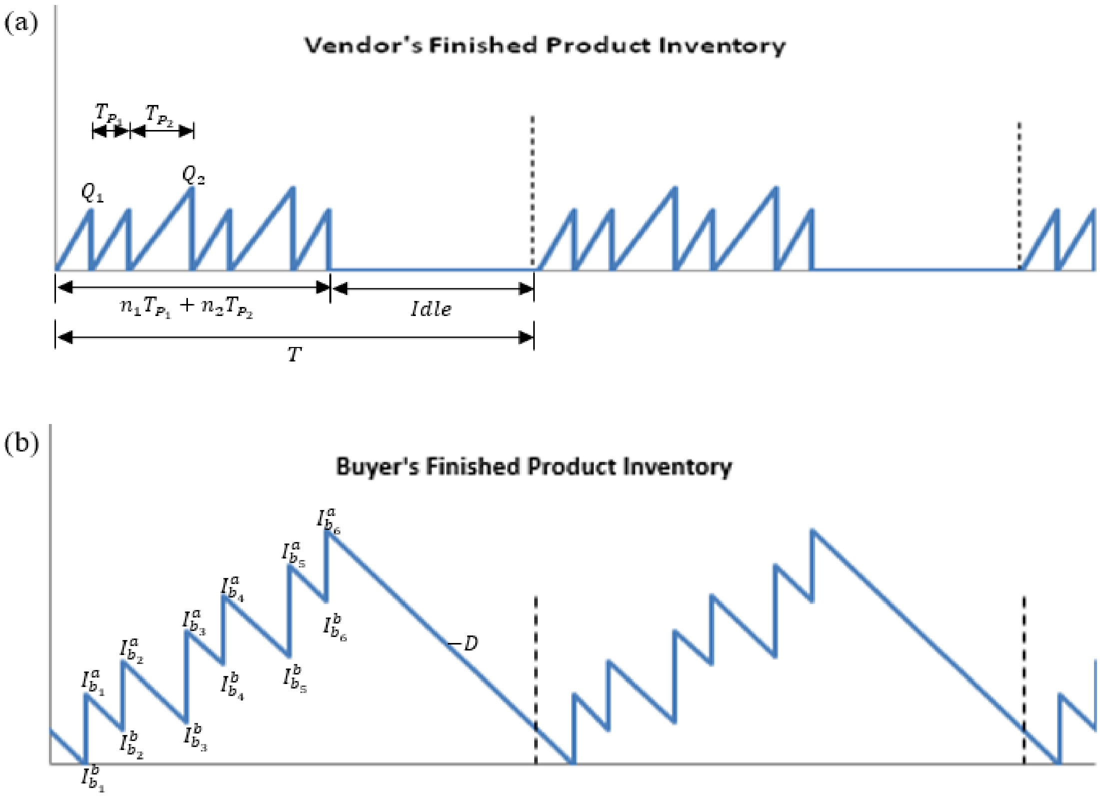
In addition, the total production uptime can be determined as a function of the production time per batch $\left(T_{P_i}\right)$ and number of production runs for product type i (n_i) and it is calculated as follows:

$$\sum_{i=1}^{2} n_i T_{P_i} = n_1 T_{P_1} + n_2 T_{P_2} = n_1 \left(\frac{d_1}{P_1\, n_1} T\right) + n_2 \left(\frac{d_2}{P_2\, n_2} T\right) = T\left(\frac{d_1}{P_1} + \frac{d_2}{P_2}\right)$$

Accordingly, the idle time is:

$$T - T\left(\frac{d_1}{P_1} + \frac{d_2}{P_2}\right) \quad = T\left(1 - \frac{d_1}{P_1} - \frac{d_2}{P_2}\right) = T\rho \tag{40}$$

Sub-figures a, b, c and d in Figure 14 show the vendor's and buyer's inventories for the sequence (1,1,2,1,2,1) where $n = 6$, $n_1 = 4$, and $n_2 = 2$.



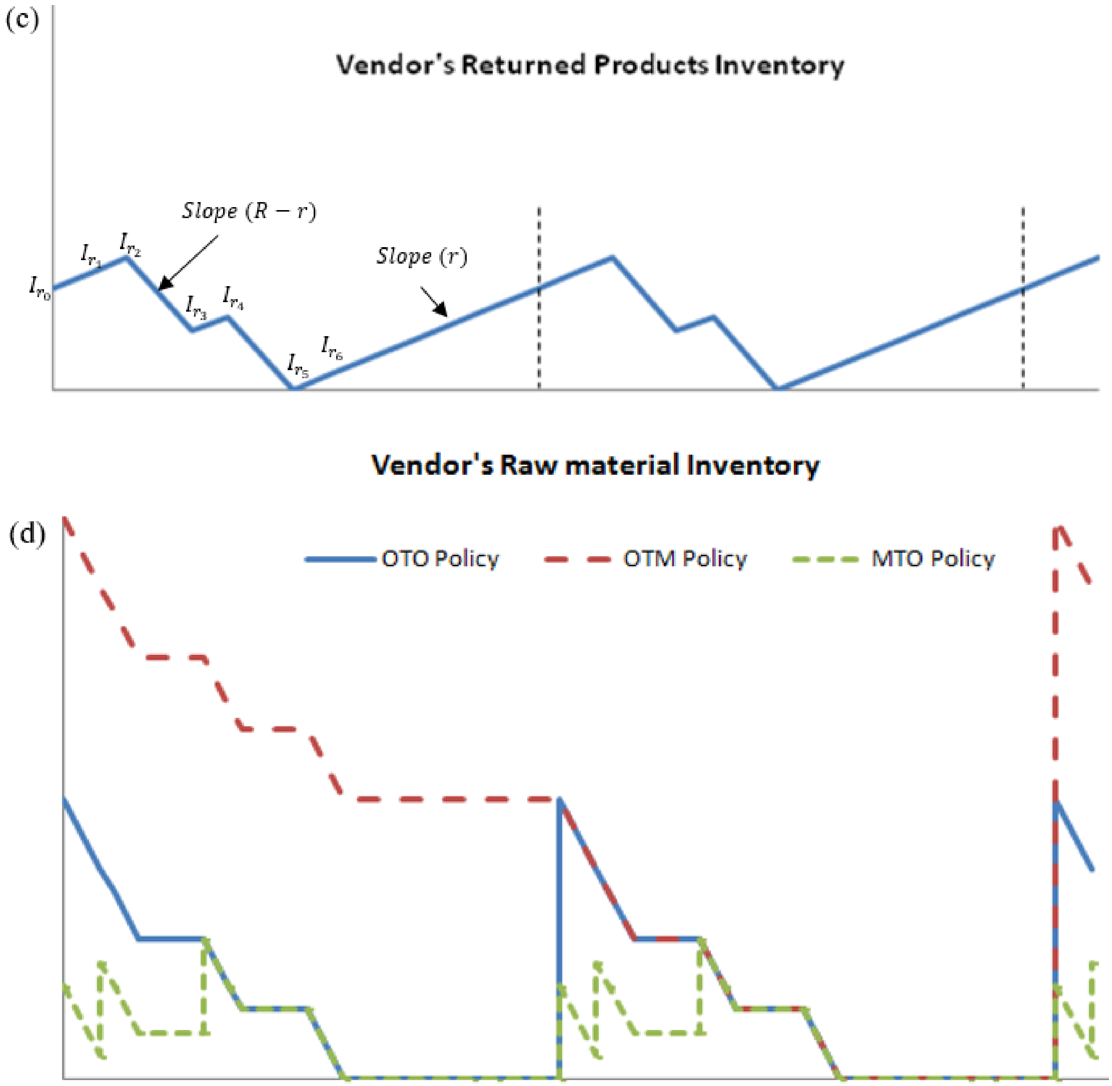


Figure 14: Vendor's and buyer's Inventory levels for $n = 6$ ($n_1 = 4, n_2 = 2$)

The objective of our model is to determine the optimal batch sizes of manufacturing (Q_1) and remanufacturing (Q_2), where those are functions of the cycle time (T), the number of those batches (n_1 and n_2), the production sequence at the vendor's stage along with the initial recovered product inventory level (I_{r0}) and initial inventory level at the buyer's stage (I_{b1}^{b}) that will minimize the long-run system wide costs consisting of the inventory holding costs of raw materials and recovered products at the vendor stage, finished products holding cost at the vendor stage, finished products holding cost at the buyer stage, setup costs of products 1 and 2, and the buyer's and vendor's ordering costs.

In the subsequent analysis, it is assumed that the number of batches for manufacturing and remanufacturing is given. Define the following binary variables:

$$x_{ij} = \begin{cases} 1 \; if \; product \; i \; is \; produced \; at \; the \; j^{th} \; production \; run, i = 1,2, and \; j = 1,2, \dots, n \\ 0 \; otherwise \end{cases}$$

These binary variables are subject to the following conditions:

$$\sum_{j=1}^{n} x_{ij} = n_i \qquad i = 1,2 \tag{41}$$

$$\sum_{i=1}^{2} x_{ij} = 1 \qquad j = 1,2, \dots, n \tag{42}$$

Constraints (41) ensures that the number of setups of product i is equal to n_i, while constraints (42) ensure that at every production run, only one type of product is produced. The following sub-sections show the detailed derivation of the inventory costs of finished products at the vendor and buyer stages, and the returned products at the vendor stage. The derivation of the raw material inventory cost is saved for last as it depends on the raw material replenishment policy in place.

4.1.1. Derivation of the vendor's finished products inventory holding and setup costs

Referring to Figure 14a, we can observe that during the production uptime portion of the cycle, equal batches of size Q_i, where possibly $Q_1 \neq Q_2$, are produced and shipped every T_{P_i} units of time until production ceases. Hence, the vendor's inventory holding cost per unit of time is given by [24]:

$$H_v = \frac{h_v}{T}\left(\frac{T^2}{2}\sum_{i=1}^{2}\frac{d_i^2}{P_i\, n_i}\right) = T.\frac{h_v}{2}\sum_{i=1}^{2}\frac{d_i^2}{P_i\, n_i} = T.RH_v \tag{43}$$

where RH_v is the relative vendor's finished products inventory holding cost per unit of time.

Next, to determine the setup cost of the manufacturing and remanufacturing process, the following binary variables are introduced:

$$y_{ij} = \begin{cases} 1 \; if \; a \; setup \; for \; product \; i \; is \; made \; at \; the \; jth \; production \; run, \\ 0 \; otherwise \end{cases}$$

Note that a setup takes place only for the first batch and when two different product types are produced during two consecutive production runs. Hence, the binary variables y_{ij} should satisfy the following conditions:

$$y_{i1} \geq x_{i1} \qquad i = 1, 2 \tag{44}$$

$$y_{ij} \geq x_{ij} - x_{i(j-1)} \qquad i = 1, 2 \;\; and\, j = 2, 3, \dots, n \tag{45}$$

The vendor's total set up cost per unit time is thus given by:

$$S_v = \frac{1}{T}.\left(\sum_{i=1}^{2} s_i \sum_{j=1}^{n} y_{ij}\right) = \frac{1}{T}.s_v \tag{46}$$

where s_i is the set up cost for product type i.

4.1.2. **Derivation of the buyer's holding and ordering costs.** In order to determine the holding cost per unit time for the finished products, we need to find the total area under the inventory graph. The following variables determine the inventory levels of the buyer before and after each production run:

$I_{b_j}^{b}$: Inventory level of buyer's finished product just before the receipt of the jth shipment.

$I_{b_j}^{a}$: Inventory level of buyer's finished product just after the receipt of the jth shipment.

Note that the size of the jth shipment and the length of the jth production run depend on the product type being processed during the jth production run. Hence, let:

T_j = length of the jth production run, which is given by

$$T_j = \sum_{i=1}^{2} T_{P_i} x_{ij} = T \sum_{i=1}^{2} t_{P_i} x_{ij} = T.RT_j \qquad j = 1,2, \dots, n \tag{47}$$

where RT_j is the relative length of the jth production run.

Similarly, define S_j to be the jth shipment size, which is given by

$$S_j = \sum_{i=1}^{2} Q_i x_{ij} = T \sum_{i=1}^{2} q_i x_{ij} = T.RS_j \qquad j = 1,2, \dots, n \tag{48}$$

where RS_j is the relative shipment size of the jth production run.

Referring to Figure 14b, we can see that:

$$\mathrm{I}_{\mathrm{b_j}}^{\mathrm{a}} = \mathrm{I}_{\mathrm{b_j}}^{\mathrm{b}} + S_j \qquad \text{for } j = 1,2,\dots,n, \text{ and}$$

$$\mathrm{I}_{\mathrm{b_j}}^{\mathrm{b}} = \mathrm{I}_{\mathrm{b_{(j-1)}}}^{\mathrm{a}} - \mathrm{D}T_j \qquad \text{for } j = 2,3,\dots,n$$

Referring back to Hariga et al. [24], the inventory level of buyer's finished product just before the receipt of the j[th] shipment is given by:

$$I_{b_j}^{b} = T\left[RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j-1} RS_k\right] = T.RI_{b_j}^{b} \qquad \text{for } j = 2,3,\dots,n \tag{49}$$

where $RI_{b_1}^{b}$ is the relative initial inventory level of the buyer's finished products and $I_{b_j}^{b}$, for j = 2, 3,…, n, is the relative inventory level for finished products just before the receipt of the j[th] shipment.

Since $I_{b_j}^{a} = I_{b_j}^{b} + S_j$, then

$$I_{b_j}^{a} = T\left[RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j} RS_k\right] = T.RI_{b_j}^{a} \qquad \text{for } j = 2,3,\dots,n \tag{50}$$

where $RI_{b_j}^{a}$, for j = 2,3,…, n, is the relative inventory level of the buyer's finished products just after the receipt of the j[th] shipment.

As per Hariga et al. [24], the buyer's finished products inventory holding cost per unit of time is given by:

$$H_b = T.\frac{h_b}{2}\left[\sum_{j=2}^{n}\left(RI_{b_{j-1}}^{a} + RI_{b_j}^{b}\right)RT_j + \left(RI_{b_n}^{a} + RI_{b_1}^{b}\right)(\rho + RT_1)\right] = \mathrm{T.R}H_b \tag{51}$$

where $\mathrm{R}H_b$ is the relative buyer's finished products inventory holding cost per unit of time.

The buyer's ordering cost per unit of time is simply given by:

$$A_b = \frac{nO_b}{T} \tag{52}$$

where O_b is the buyer's ordering cost per order.

4.1.3. **Derivation of the vendor's returned products holding cost.** Referring to Figure 14c, it can be seen that the inventory of the returned products increases at a rate of r units per unit of time when product type 1 is being manufactured during the jth production run, and decreases at a net rate of $(R-r)$ units per unit of time when product type 2 is being manufactured during the jth production run. Let:

I_{r_0}: Initial stock of returned products at the begining of the production cycle

I_{r_j}: Inventory level of returned products just after the jth production run

Referring back to Hariga et al. [24] model, the inventory level of returned products just after the j[th] production run is given by:

$$I_{r_j} = T\left[RI_{r_0} - (R-r)t_{P_2}\sum_{k=1}^{j} x_{2k} + rt_{P_1}\sum_{k=1}^{j} x_{1k}\right] = T.RI_{r_j} \tag{53}$$

where RI_{r_0} is the relative initial inventory level for returned products and I_{r_j} , for j = 1, 2,…, n, is the relative inventory level of returned products just after the j[th] production run.

Based on the model developed by Hariga et al. [24], the vendor's returned products inventory holding cost per unit of time is given by:

$$H_r = T.\frac{h_r}{2}\left[\sum_{j=1}^{n}\left(RI_{r_{j-1}} + RI_{r_j}\right)RT_j + \left(RI_{r_n} + RI_{r_0}\right)\rho\right] = T.RH_r \tag{54}$$

where RH_r is the relative vendor's returned products inventory holding cost per unit of time.

4.1.4. **Derivation of the vendor's raw material holding and ordering costs.** The vendor's raw material holding cost can be determined by finding the area under the inventory profile of raw material. The level of raw material inventory decreases by a rate of P_1 per unit of time when product type 1 is produced during the jth production run and stays constant when product type 2 is produced or during the idle time. On the other hand, the level of inventory increases by Q_{rm_j} only when ordering a raw material from the supplier where the lead time is taken to be negligible and where Q_{rm_j} denotes the quantity of raw material received at the beginning of the jth production run. Because of the sequencing issue in production, Q_{rm_j} is ordered and received when it is needed. This means that if the production sequence is for instance (2,2,1,1,2), then we will order and receive the raw material shipment just before the 3rd production run rather than at the beginning of the cycle. To decide when to order the Q_{rm_j} shipments, we need to introduce the following binary variable:

$$a_j = \begin{cases} 1 \; if \; a \; raw \; material \; order \; is \; placed \; at \; the \; beginning \; of \; the \; j^{th} \; production \; run \\ 0 \; otherwise \end{cases}$$

As mentioned earlier, the different ordering policies affect the levels of inventory and accordingly the inventory holding cost. Therefore, a model will be derived for each of these three policies again.

4.1.4.1. ***One-To-One policy (OTO).*** Following this policy, the vendor orders one raw material shipment of size Q_{rm_j} that is enough to cover the production of product type 1 for one whole cycle. Figure 15 shows the raw material inventory profile under this policy for the same example that was mentioned earlier (i.e. sequence 1,1,2,1,2,1):

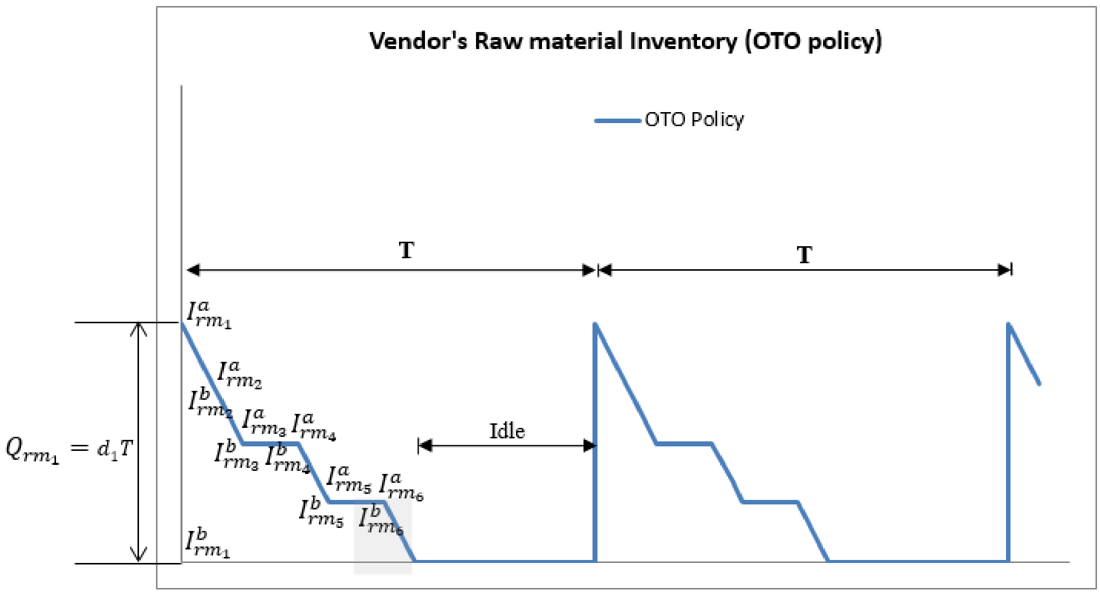


Figure 15: Vendor's raw material inventory operating under OTO policy

Referring to Figure 15, we can see that the maximum level of inventory is Q_{rm_1}. This level decreases by a rate of P_1 when producing product type 1 and stays constant when producing product type 2. In addition, the inventory reaches zero by the time when production ceases. In this example, because product type 1 is being produced in the first production run, a raw material order of size d_1T is placed at the beginning of the cycle. Let

Q_{rm_j}: Quantity of raw material received at the beginning of the jth production run

$I^b_{rm_j}$: raw material inventory level at the beginning of the jth production run, before receiving Q_{rm_j} (j $= 1, 2, \dots,$ n)

$I^a_{rm_j}$: raw material inventory level at the beginning of the jth production run, after receiving Q_{rm_j} (j $= 1, 2, \dots,$ n)

Recall the definition of the binary variable a_j where

$$a_j = \begin{cases} 1 \; if \; a \; raw \; material \; order \; is \; placed \; at \; the \; beginning \; of \; the \; j^{th} \; production \; run \\ 0 \; otherwise \end{cases}$$

Now, an order of quantity Q_{rm_j} is placed only when a_j is equal to one, which can be mathematically expressed as:

$$Q_{rm_j} \leq d_1 T a_j \text{ , or alternatively } q_{rm_j} \leq d_1 a_j \qquad (\text{j} = 1, 2, \dots, \text{n}) \qquad (55)$$

Under the OTO policy, number of raw material orders within a cycle ($\sum_{j=1}^{n} a_j$) needs to be equal to 1, Hence

$$\sum_{j=1}^{n} a_j = 1, \qquad (\text{j} = 1, 2, \dots, \text{n}) \qquad (56)$$

It is easy to see that the inventory level after receiving Q_{rm_j} is given by:

$$I^a_{rm_j} = I^b_{rm_j} + Q_{rm_j}$$

And in the case where no raw material shipment is received at the beginning of the jth production run ($Q_{rm_j} = 0$), the inventory levels at the beginning of the jth production run are equal:

$$I^b_{rm_j} = I^a_{rm_j}$$

Referring back to Figure 15, we have

$I^{b}_{rm_1} = 0$

$$I^{b}_{rm_2} = I^{a}_{rm_1} - P_1 T_{P_1} x_{11} = \left(I^{b}_{rm_1} + \mathrm{Q}_{\mathrm{rm}_1}\right) - P_1 T_{P_1} x_{11} = \mathrm{Q}_{\mathrm{rm}_1} - P_1 T_{P_1} x_{11}$$

$$I^{b}_{rm_3} = I^{a}_{rm_2} - P_1 T_{P_1} x_{12} = \left(I^{b}_{rm_2} + \mathrm{Q}_{\mathrm{rm}_2}\right) - P_1 T_{P_1} x_{12}$$

$$= \left(\mathrm{Q}_{\mathrm{rm}_1} - P_1 T_{P_1} x_{11} + \mathrm{Q}_{\mathrm{rm}_2}\right) - P_1 T_{P_1} x_{12}$$

$$= \left(\mathrm{Q}_{\mathrm{rm}_1} + \mathrm{Q}_{\mathrm{rm}_2}\right) - P_1 T_{P_1} (x_{11} + x_{12})$$

$$I^{b}_{rm_4} = I^{a}_{rm_3} - P_1 T_{P_1} x_{13} = \left(I^{b}_{rm_3} + \mathrm{Q}_{\mathrm{rm}_3}\right) - P_1 T_{P_1} x_{13}$$

$$= \left(\left(\mathrm{Q}_{\mathrm{rm}_1} + \mathrm{Q}_{\mathrm{rm}_2}\right) - P_1 T_{P_1} (x_{11} + x_{12}) + \mathrm{Q}_{\mathrm{rm}_3}\right) - P_1 T_{P_1} x_{13}$$

$$= \left(\mathrm{Q}_{\mathrm{rm}_1} + \mathrm{Q}_{\mathrm{rm}_2} + \mathrm{Q}_{\mathrm{rm}_3}\right) - P_1 T_{P_1} (x_{11} + x_{12} + x_{13})$$

In general,

$$I^{b}_{rm_j} = \sum_{k=1}^{j-1} \mathrm{Q}_{\mathrm{rm}_\mathrm{k}} - P_1 T_{P_1} \sum_{k=1}^{j-1} x_{1k} = T \sum_{k=1}^{j-1} \mathrm{q}_{\mathrm{rm}_\mathrm{k}} - P_1 T.\, t_{P_1} \sum_{k=1}^{j-1} x_{1k}$$

$$= T \left[\sum_{k=1}^{j-1} \mathrm{q}_{\mathrm{rm}_\mathrm{k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \right] = T.\, RI^{b}_{rm_j} \qquad (57)$$

where $RI^{b}_{rm_j}$, for j = 2,3,…, n, is the relative inventory level for raw material at the beginning of the j[th] production run and before receiving Q_{rm_j}, and that q_{rm_k} is the relative raw material shipment quantity at the beginning of the k[th] production run for $k = 1,2,3, \dots, j-1$.

Note that

$$RI^{b}_{rm_1} = RI^{b}_{rm_{n+1}} = 0 \qquad (58)$$

Similarly, since $I^{a}_{rm_j} = I^{b}_{rm_j} + Q_{rm_j}$. Then,

$$I^{a}_{rm_j} = \sum_{k=1}^{j} Q_{rm_k} - P_1 T_{P_1} \sum_{k=1}^{j-1} x_{1k} = T \sum_{k=1}^{j} q_{rm_k} - P_1 T.\, t_{P_1} \sum_{k=1}^{j-1} x_{1k}$$

$$= T\left[\sum_{k=1}^{j} \mathrm{q_{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k}\right] = T.RI^{a}_{rm_j} \tag{59}$$

where $RI^{a}_{rm_j}$, for j = 2,3,…, n, is the relative inventory level of raw material at the beginning of the jth production run and after receiving Q_{rm_j}.

Note that

$$RI^{a}_{rm_1} = q_{rm_1} \tag{60}$$

$$RI^{a}_{rm_{n+1}} = 0 \tag{61}$$

The sum of the trapezoids corresponds to the area under the inventory graph. This area is given by:

$$A = \frac{1}{2}\sum_{j=1}^{n-1}\left(I^{a}_{rm_j} + I^{b}_{rm_{j+1}}\right)T_j + \frac{1}{2}I^{a}_{rm_n}T_n$$

$$= \frac{1}{2}\sum_{j=1}^{n-1}\left(T.RI^{a}_{rm_j} + T.RI^{b}_{rm_{j+1}}\right)T.RT_j + \frac{1}{2}T.RI^{a}_{rm_n}T.RT_n$$

$$= \frac{T^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a}_{rm_j} + RI^{b}_{rm_{j+1}}\right)RT_j + RI^{a}_{rm_n}RT_n\right]$$

The vendor’s raw materials inventory holding cost per unit of time is thus given by:

$$H_{rm} = \frac{h_{rm}}{T}[A] = \frac{h_{rm}}{T}\left[\frac{T^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a}_{rm_j} + RI^{b}_{rm_{j+1}}\right)RT_j + RI^{a}_{rm_n}RT_n\right]\right]$$

$$= T.\frac{h_{rm}}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a}_{rm_j} + RI^{b}_{rm_{j+1}}\right)RT_j + RI^{a}_{rm_n}RT_n\right] = T.RH_{rm} \tag{62}$$

where RH_{rm} is the relative vendor’s raw materials inventory holding cost per unit of time.

The vendor’s raw materials ordering cost per unit of time is given by:

$$A_v = \frac{O_v}{T} \tag{63}$$

4.1.4.2. ***One-To-Multi policy (OTM)***. In this policy, the vendor orders one raw material shipment of size Q_{rm_j} that is enough to produce type 1 product to satisfy the $(D - r)$ demand of multiple cycles (u). Figure 16 shows the raw material inventory profile under this policy for u = 2, where the production sequence is (1,1,2,1,2,1).

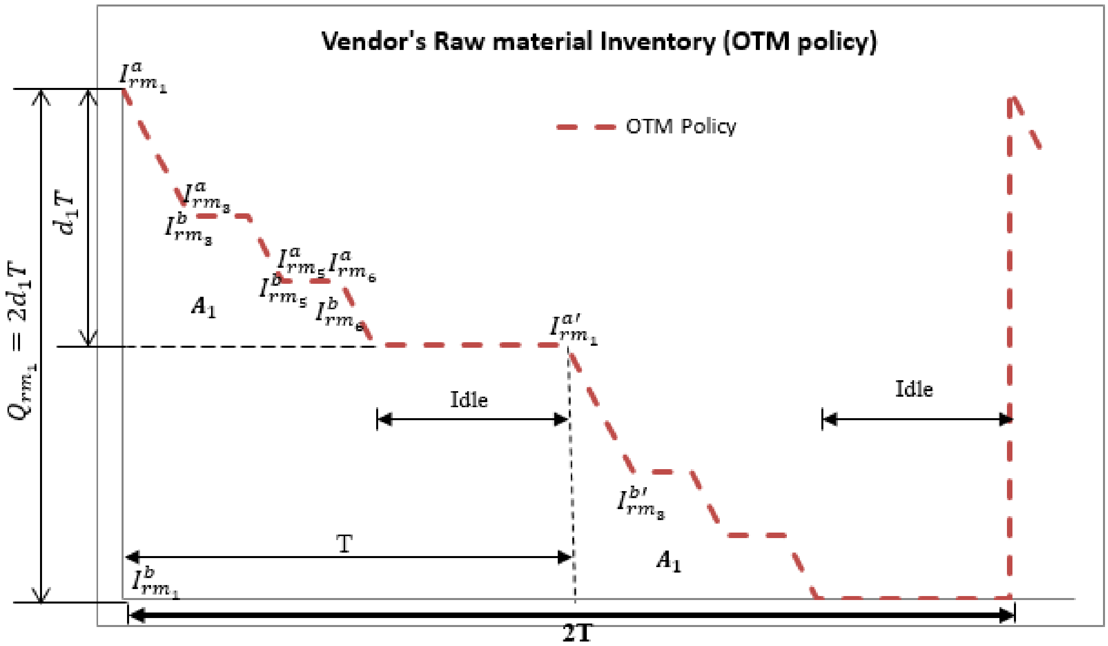


Figure 16: Vendor's raw material inventory operating under OTM policy (u=2)

Referring to Figure 16, we can see that the maximum level of inventory is Q_{rm_j}. This level decreases by a rate of P_1 when producing product type 1 and stays constant when producing product type 2 or by the time when production ceases. The inventory drops by (d_1T) after the production up time in every cycle and reaches zero during the idle time of the last cycle regardless of any sequence. Accordingly, since Q_{rm_j} is enough to cover the production for two consecutive cycles $(2d_1T)$, the level of inventory when production ceases during the first cycle is half Q_{rm_j}. This level is exactly equal to the inventory level at the beginning of the second cycle.

Now, an order of quantity Q_{rm_j} is placed only when a_j is equal to one, which can be mathematically expressed as:

$$Q_{rm_j} \leq ud_1Ta_j\,, \quad \text{or equivalently,} \quad q_{rm_j} \leq ud_1a_j \quad (\text{j} = 1, 2, \dots, \text{n}) \quad (64)$$

In general, number of raw material orders within the first cycle ($\sum_{j=1}^{n} a_j$) cannot be greater than 1, Hence

$$\sum_{j=1}^{n} a_j = 1, \qquad (\text{j} = 1, 2, \dots, \text{n}) \qquad (65)$$

Furthermore, to find the area under this inventory profile, consider A_1 of this inventory profile as a single OTO raw material profile in which the raw material shipment is enough to satisfy d_1T (the area under the curve in the 2nd cycle). As was discussed in section 1.5.1, we know that

$$q'_{rm_j} \leq d_1a_j$$

$$RI^{b\prime}_{rm_j} = \sum_{k=1}^{j-1} q'_{rm_k} - P_1t_{P_1} \sum_{k=1}^{j-1} x_{1k}$$

$$RI^{a\prime}_{rm_j} = \sum_{k=1}^{j} q'_{rm_k} - P_1t_{P_1} \sum_{k=1}^{j-1} x_{1k}$$

where q'_{rm_j} , $RI^{b\prime}_{rm_j}$ and $RI^{a\prime}_{rm_j}$ represents q_{rm_j} , $RI^{b}_{rm_j}$ and $RI^{a}_{rm_j}$ in section 1.5.1, respectively. Then,

$$A_1 = \frac{T^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a\prime}_{rm_j} + RI^{b\prime}_{rm_{j+1}}\right)RT_j + RI^{a\prime}_{rm_n}RT_n\right]$$

And that

$$RI^{a\prime}_{rm_1} = q'_{rm_1} \qquad (66)$$

Then, total inventory is calculated as follows:

$$A = \underbrace{A_1 + d_1\text{T}.\text{T}}_{1^{st}cycle} + \underbrace{A_1}_{2^{nd}cycle}$$

$$= 2A_1 + d_1\text{T}^2$$

When we order raw materials to cover 3 cycles (u =3), then

$$A = \underbrace{A_1 + 2d_1\mathrm{T.T}}_{1^{st}cycle} + \underbrace{A_1 + d_1\mathrm{T.T}}_{2^{nd}cycle} + \underbrace{A_1}_{3^{rd}cycle}$$

$$= 3A_1 + d_1\mathrm{T}^2 + 2d_1\mathrm{T}^2$$

$$= 3A_1 + d_1\mathrm{T}^2(1+2)$$

When we order raw materials to cover 4 cycles (u =4), then

$$A = \underbrace{A_1 + 3d_1\mathrm{T.T}}_{1^{st}cycle} + \underbrace{A_1 + 2d_1\mathrm{T.T}}_{2^{nd}cycle} + \underbrace{A_1 + d_1\mathrm{T.T}}_{3^{rd}cycle} + \underbrace{A_1}_{4^{th}cycle}$$

$$= 4A_1 + d_1\mathrm{T}^2 + 2d_1\mathrm{T}^2 + 3d_1\mathrm{T}^2$$

$$= 4A_1 + d_1\mathrm{T}^2(1+2+3)$$

In general,

$$A = uA_1 + d_1\mathrm{T}^2\big(1 + 2 + .. + (\mathrm{u}-1)\big)$$

$$= uA_1 + d_1\mathrm{T}^2\left(\frac{\mathrm{u}(\mathrm{u}-1)}{2}\right)$$

$$= u\left(\frac{T^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a\prime}_{rm_j} + RI^{b\prime}_{rm_{j+1}}\right)RT_j + RI^{a\prime}_{rm_n}RT_n\right]\right) + d_1\mathrm{T}^2\left(\frac{\mathrm{u}(\mathrm{u}-1)}{2}\right)$$

$$= \frac{uT^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a\prime}_{rm_j} + RI^{b\prime}_{rm_{j+1}}\right)RT_j + RI^{a\prime}_{rm_n}RT_n + d_1(\mathrm{u}-1)\right]$$

Accordingly, the inventory holding cost of raw material per unit of time is given by:

$$H^u_{rm} = \frac{h_{rm}}{uT}\left[\frac{uT^2}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a\prime}_{rm_j} + RI^{b\prime}_{rm_{j+1}}\right)RT_j + RI^{a\prime}_{rm_n}RT_n + d_1(\mathrm{u}-1)\right]\right]$$

$$= T.\frac{h_{rm}}{2}\left[\sum_{j=1}^{n-1}\left(RI^{a\prime}_{rm_j} + RI^{b\prime}_{rm_{j+1}}\right)RT_j + RI^{a\prime}_{rm_n}RT_n + d_1(\mathrm{u}-1)\right]$$

$$= T.RH_{rm}^{u} \tag{67}$$

where RH_{rm}^{u} is the relative raw material inventory holding cost per unit of time for u cycles.

Additionally, since the vendor orders only once every u cycles, the vendor's raw materials ordering cost per unit of time is then given by:

$$A_v = \frac{O_v}{uT} \tag{68}$$

where O_v is the vendor's ordering cost of raw materials per order.

4.1.4.3. ***Multi-To-One (MTO).*** In this policy, the vendor orders multiple raw material shipments of size Q_{rm_j} each during one cycle to satisfy the production of type 1 product in one cycle. As such, Q_{rm_j} would be enough to produce a fraction of d_1 in one cycle (that is, a fraction of d_1T). Note that Q_{rm_j} is ordered and received during the production uptime before producing type 1 product if raw material inventory level just before the production run is not enough to produce one full batch of product type 1. Figure 17 shows the raw material inventory profile under this policy for the same example that was mentioned earlier where three raw material shipments of equal sizes are ordered in one cycle.

Recall the definitions of Q_{rm_j} , $I_{rm_j}^{b}$, $I_{rm_j}^{a}$ and the binary variable a_j. Now, a raw material shipment will be received only when a_j is equal to one, which can be mathematically expressed as:

$$Q_{rm_j} \leq d_1 T a_j \text{ , or equivalently, } q_{rm_j} \leq d_1 a_j \qquad (\text{j} = 1, 2, \dots, \text{n})$$

In general, number of raw material orders within a cycle ($\sum_{j=1}^{n} a_j$) cannot be greater than the number of production batches for product type 1 because Q_{rm_j} will no longer be sufficient to produce one lot size (Q_1). In other words, minimum Q_{rm_j} must be equal to Q_1 when ordering a raw material shipment. Hence,

$$Q_{rm_j} \geq Q_1 a_j, \text{ or equivalently, } q_{rm_j} \geq \frac{d_1}{n_1} a_j, \qquad (\text{j} = 1, 2, \dots, \text{n})$$

From the previous two equations, we can see that

$$\frac{d_1}{n_1}a_j \leq q_{rm_j} \leq d_1 a_j \quad , \qquad (\mathrm{j} = 1, 2, \dots, \mathrm{n}) \tag{69}$$

Hence,

$$1 \leq \sum_{j=1}^{n} a_j \leq n_1 \tag{70}$$

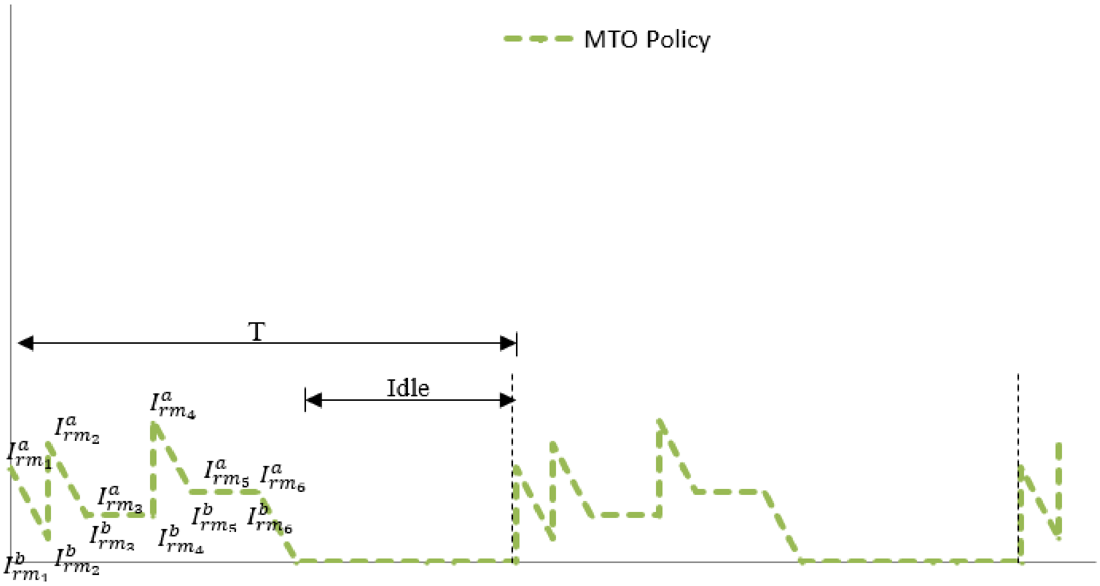


Figure 17: Vendor's raw material inventory operating under MTO policy

Recall from Eq. (57) to Eq. (61) that

$$I^b_{rm_j} = T\left[\sum_{k=1}^{j-1} \mathrm{q_{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k}\right] = T.RI^b_{rm_j}$$

$$RI^b_{rm_1} = RI^b_{rm_{n+1}} = 0$$

$$I^a_{rm_j} = T\left[\sum_{k=1}^{j} \mathrm{q_{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k}\right] = T.RI^a_{rm_j}$$

$$RI^a_{rm_1} = \mathrm{Q_{rm_1}}$$

Then

$$H^m_{rm} = \frac{h_{rm}}{2}\left[\sum_{j=1}^{n-1}\left(RI^a_{rm_j} + RI^b_{rm_{j+1}}\right)RT_j + RI^a_{rm_n}RT_n\right] = T.RH^m_{rm} \tag{71}$$

where RH_{rm}^{m} is the relative vendor's raw materials inventory holding cost per unit of time. The vendor's raw materials ordering cost per unit of time is given by:

$$A_v = \frac{O_v}{T} . \sum_{j=1}^{n} a_j \quad (72)$$

where O_v is the vendor's ordering cost of raw materials per order.

4.2. Optimization Model

4.2.1. **Three-Stage Closed Loop Supply Chain with CS partnership operating with OTO ordering policy of raw materials (3S-CLSC-CS-OTO).** As mentioned earlier, the total long-run average system wide cost is comprised of the following:

- The setup cost at the vendor stage (S_v).
- The inventory holding cost of raw materials at the vendor stage (H_{rm}).
- The inventory holding cost of returned products at the vendor stage (H_r).
- The inventory holding cost of finished products at the vendor stage (H_v).
- The inventory holding cost of finished products at the buyer stage (H_b).
- The vendor's ordering cost of raw materials (A_v).
- The buyer's ordering cost of finished products (A_b).

Using the derived equations of these cost components, the total system cost per unit of time, K, is mathematically expressed as:

$$K(n_1, n_2) = A_v + A_b + S_v + T(RH_{rm} + RH_r + RH_v + RH_b)$$

$$= \frac{O_v + nO_b + s_v}{T} + T(RH_{rm} + RH_r + RH_v + RH_b) \quad (73)$$

Finally, the optimization model of the three-stage closed loop supply chain with CS partnership for given production frequencies (n_1, n_2) with OTO raw material ordering policy can be stated as follows:

Min $K(n_1, n_2) = \frac{O_v + nO_b + s_v}{T} + T(RH_{rm} + RH_r + RH_v + RH_b)$

s.t

$$\sum_{j=1}^{n} x_{ij} = n_i \qquad i = 1,2$$

$$\sum_{i=1}^{2} x_{ij} = 1 \qquad j = 1,2,\dots,n$$

$$\sum_{j=1}^{n} a_j = 1$$

$$q_{rm_j} \leq d_1 a_j \qquad j = 1,2,\dots,n$$

$$y_{i1} \geq x_{i1} \qquad i = 1,2$$

$$y_{ij} \geq x_{ij} - x_{i(j-1)} \qquad i = 1,2 \;\; and\; j = 2,3,\dots,n$$

$$RT_j = \sum_{i=1}^{2} t_{P_i} x_{ij} \qquad j = 1,2,\dots,n \qquad (74)$$

$$RS_j = \sum_{i=1}^{2} q_i x_{ij} \qquad j = 1,2,\dots,n \qquad (75)$$

$$s_v = \sum_{i=1}^{2} s_i \sum_{j=1}^{n} y_{ij}$$

$$RI_{b_j}^{b} = RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j-1} RS_k \qquad j = 2,3,\dots,n \qquad (76)$$

$$RI_{b_j}^{a} = RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j} RS_k \qquad j = 2,3,\dots,n \qquad (77)$$

$$RI_{b_1}^{a} = RI_{b_1}^{b} + RS_j$$

$$RI_{r_j} = RI_{r_0} - (R-r)t_{P_2}\sum_{k=1}^{j} x_{2k} + rt_{P_1}\sum_{k=1}^{j} x_{1k} \qquad j = 1,2,\dots,n \qquad (78)$$

$$RI_{rm_j}^{b} = \sum_{k=1}^{j-1} \mathrm{q_{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3,\dots,n \tag{79}$$

$$RI_{rm_j}^{a} = \sum_{k=1}^{j} \mathrm{q_{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3,\dots,n \tag{80}$$

$$RI_{rm_1}^{b} = 0$$

$$RI_{rm_1}^{a} = q_{rm_1}$$

$$RH_v = \frac{h_v}{2} \sum_{i=1}^{2} \frac{d_i^2}{P_i\, n_i} \tag{81}$$

$$\mathrm{R}H_b = \frac{h_b}{2}\left[\sum_{j=2}^{n} \left(RI_{b_{j-1}}^{a} + RI_{b_j}^{b}\right) RT_j + \left(RI_{b_n}^{a} + RI_{b_1}^{b}\right)(\rho + RT_1)\right] \tag{82}$$

$$RH_r = \frac{h_r}{2}\left[\sum_{j=1}^{n} \left(RI_{r_{j-1}} + RI_{r_j}\right) RT_j + \left(RI_{r_n} + RI_{r_0}\right)\rho\right] \tag{83}$$

$$RH_{rm} = \frac{h_{rm}}{2}\left[\sum_{j=1}^{n-1} \left(RI_{rm_j}^{a} + RI_{rm_{j+1}}^{b}\right) RT_j + RI_{rm_n}^{a} RT_n\right] \tag{84}$$

$$RI_{r_{j-1}} \geq (R - r) t_{P_2} x_{2j} \qquad j = 1,2,\dots,n \tag{85}$$

$$RI_{r_n} \geq 0 \tag{86}$$

$$RI_{b_j}^{b} \geq 0 \qquad j = 1,2,\dots,n \tag{87}$$

$$RI_{rm_j}^{a} \geq P_1 t_{P_1} x_{1j} \qquad j = 1,2,\dots,n \tag{88}$$

$$RI_{rm_j}^{b} \geq 0 \qquad j = 1,2,\dots,n \tag{89}$$

$$RI_{rm_j}^{a} \geq 0 \qquad j = 1,2,\dots,n \tag{90}$$

$$q_{rm_j} \geq 0 \qquad j = 1,2,\dots,n \tag{91}$$

$x_{ij}, y_{ij} \in \{0,1\} \qquad i = 1,2 \; and \; j = 1,2,\dots,n \qquad (92)$

$a_j \in \{0,1\} \qquad j = 1,2,\dots,n \qquad (93)$

Constraint sets (85) and (86) ensure that the inventory of the returned products is enough to avoid shortages during the remanufacturing process at the j[th] production run. Constraints (87) guarantee that the buyer always has enough finished products to satisfy the demand, which eliminates the shortages in the buyer's facility of finished products. Constraints (88), (89) and (90) stipulate that shortages are not allowed in the raw material inventory of the vendor.

4.2.2. **Three-Stage Closed Loop Supply Chain with CS partnership operating with OTM ordering policy of raw materials (3S-CLSC-CS-OTM).** Similar to the previous model, the total long-run average system wide cost consists of the following cost components:

- The setup cost at the vendor stage (S_v).
- The inventory holding cost of raw materials at the vendor stage (H_{rm}^u).
- The inventory holding cost of returned products at the vendor stage (H_r).
- The inventory holding cost of finished products at the vendor stage (H_v).
- The inventory holding cost of finished products at the buyer stage (H_b).
- The buyer's ordering cost of finished products (A_b).
- The vendor's ordering cost of raw materials (A_v).

Using the derived equations of these cost components, the total system cost per unit of time, K, is mathematically expressed as:

$$K(n_1, n_2, u) = A_v + A_b + S_v + T(RH_{rm}^u + RH_r + RH_v + RH_b)$$

$$= \frac{\frac{O_v}{u} + nO_b + s_v}{T} + T(RH_{rm}^u + RH_r + RH_v + RH_b) \qquad (94)$$

Finally, the optimization model of the three-stage closed loop supply chain with CS partnership for given production frequencies (n_1, n_2) with OTM raw material ordering policy can be stated as follows:

$$\text{Min} \quad K(n_1, n_2, u) = \frac{\frac{O_v}{u} + nO_b + s_v}{T} + T(RH_{rm}^u + RH_r + RH_v + RH_b)$$

s.t

$$\sum_{j=1}^{n} x_{ij} = n_i \qquad i = 1,2$$

$$\sum_{i=1}^{2} x_{ij} = 1 \qquad j = 1,2,\dots,n$$

$$\sum_{j=1}^{n} a_j = 1$$

$$q'_{rm_j} \leq d_1 a_j \qquad j = 1,2,\dots,n$$

$$y_{i1} \geq x_{i1} \qquad i = 1,2$$

$$y_{ij} \geq x_{ij} - x_{i(j-1)} \qquad i = 1,2 \;\; and\; j = 2,3,\dots,n$$

$$RT_j = \sum_{i=1}^{2} t_{P_i} x_{ij} \qquad j = 1,2,\dots,n$$

$$RS_j = \sum_{i=1}^{2} q_i x_{ij} \qquad j = 1,2,\dots,n$$

$$s_v = \sum_{i=1}^{2} s_i \sum_{j=1}^{n} y_{ij}$$

$$RI_{b_j}^{b} = RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j-1} RS_k \qquad j = 2,3,\dots,n$$

$$RI_{b_j}^{a} = RI_{b_1}^{b} - D\sum_{k=2}^{j} RT_k + \sum_{k=1}^{j} RS_k \qquad j = 2,3,\dots,n$$

$$RI_{b_1}^{a} = RI_{b_1}^{b} + RS_j$$

$$RI_{r_j} = RI_{r_0} - (R-r)t_{P_2}\sum_{k=1}^{j} x_{2k} + rt_{P_1}\sum_{k=1}^{j} x_{1k} \qquad j = 1,2,\dots,n$$

$$RI_{rm_j}^{b\prime} = \sum_{k=1}^{j-1} q'_{rm_k} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3,\dots,n \tag{95}$$

$$RI_{rm_j}^{a\prime} = \sum_{k=1}^{j} q'_{rm_k} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3,\dots,n \tag{96}$$

$$RI_{rm_1}^{b\prime} = 0$$

$$RI_{rm_1}^{a\prime} = q'_{rm_1}$$

$$RH_v = \frac{h_v}{2} \sum_{i=1}^{2} \frac{d_i^2}{P_i\, n_i}$$

$$\mathrm{R}H_b = \frac{h_b}{2} \left[\sum_{j=2}^{n} \left(RI_{b_{j-1}}^{a} + RI_{b_j}^{b} \right) RT_j + \left(RI_{b_n}^{a} + RI_{b_1}^{b} \right)(\rho + RT_1) \right]$$

$$RH_r = \frac{h_r}{2} \left[\sum_{j=1}^{n} \left(RI_{r_{j-1}} + RI_{r_j} \right) RT_j + \left(RI_{r_n} + RI_{r_0} \right) \rho \right]$$

$$RH_{rm}^{u} = \frac{h_{rm}}{2} \left[\sum_{j=1}^{n-1} \left(RI_{rm_j}^{a\prime} + RI_{rm_{j+1}}^{b\prime} \right) RT_j + RI_{rm_n}^{a\prime} RT_n + d_1(\mathrm{u} - 1) \right] \tag{97}$$

$$RI_{r_{j-1}} \geq (R - r) t_{P_2} x_{2j} \qquad j = 1,2,\dots,n$$

$$RI_{r_n} \geq 0$$

$$RI_{b_j}^{b} \geq 0 \qquad j = 1,2,\dots,n$$

$$RI_{rm_j}^{a\prime} \geq P_1 t_{P_1} x_{1j} \qquad j = 1,2,\dots,n$$

$$RI_{rm_j}^{b\prime} \geq 0 \qquad j = 1,2,\dots,n$$

$$RI_{rm_j}^{a\prime} \geq 0 \qquad j = 1,2,\dots,n$$

$$q'_{rm_j} \geq 0 \qquad j = 1,2,\dots,n$$

$x_{ij}\ , y_{ij}\ \in \{0,1\} \qquad i = 1, 2\ and\ \ j = 1,2, \dots, n$

$a_j\ \ \in \{0,1\} \qquad j = 1,2, \dots, n$

$u\ \ \in N^+$ (98)

4.2.3. **Three-Stage Closed Loop Supply Chain with CS partnership operating with MTO ordering policy of raw materials (3S-CLSC-CS-MTO).** As for the previous two models, the total long-run average system wide cost is composed of the following:

- The setup cost at the vendor stage (S_v).
- The inventory holding cost of raw materials at the vendor stage (H_{rm}^m).
- The inventory holding cost of returned products at the vendor stage (H_r).
- The inventory holding cost of finished products at the vendor stage (H_v).
- The inventory holding cost of finished products at the buyer stage (H_b).
- The buyer's ordering cost of finished products (A_b).
- The vendor's ordering cost of raw materials (A_v).

Using the derived equations of these cost components, the total system cost per unit of time, K, is mathematically expressed as:

$$K(n_1, n_2) = \ A_v + A_b + S_v + T(RH_{rm} + RH_r + RH_v + RH_b)$$

$$= \ \frac{O_v \sum_{j=1}^{n} a_j + nO_b + s_v}{T} + T(RH_{rm} + RH_r + RH_v + RH_b) \qquad (99)$$

Finally, the optimization model of the three stage closed loop supply chain with CS partnership for given production frequencies (n_1, n_2) with MTO raw material ordering policy can be stated as follows:

Min $\ K(n_1, n_2) = \ \frac{O_v \sum_{j=1}^{n} a_j + nO_b + s_v}{T} + T(RH_{rm}^m + RH_r + RH_v + RH_b)$

s.t

$$\sum_{j=1}^{n} x_{ij} = \ n_i \qquad i = 1,2$$

$$\sum_{i=1}^{2} x_{ij} = 1 \qquad j = 1,2, \dots, n$$

$$\frac{d_1}{n_1} a_j \leq q_{rm_j} \leq d_1 a_j \qquad j = 1, 2, \dots, n$$

$$1 \leq \sum_{j=1}^{n} a_j \leq n_1$$

$$y_{i1} \geq x_{i1} \qquad i = 1, 2$$

$$y_{ij} \geq x_{ij} - x_{i(j-1)} \qquad i = 1, 2 \;\; and\; j = 2, 3, \dots, n$$

$$RT_j = \sum_{i=1}^{2} t_{P_i} x_{ij} \qquad j = 1,2, \dots, n$$

$$RS_j = \sum_{i=1}^{2} q_i x_{ij} \qquad j = 1,2, \dots, n$$

$$s_v = \sum_{i=1}^{2} s_i \sum_{j=1}^{n} y_{ij}$$

$$RI_{b_j}^{b} = RI_{b_1}^{b} - D \sum_{k=2}^{j} RT_k + \sum_{k=1}^{j-1} RS_k \qquad j = 2,3, \dots, n$$

$$RI_{b_j}^{a} = RI_{b_1}^{b} - D \sum_{k=2}^{j} RT_k + \sum_{k=1}^{j} RS_k \qquad j = 2,3, \dots, n$$

$$RI_{b_1}^{a} = RI_{b_1}^{b} + RS_j$$

$$RI_{r_j} = RI_{r_0} - (R - r) t_{P_2} \textstyle\sum_{k=1}^{j} x_{2k} + r t_{P_1} \textstyle\sum_{k=1}^{j} x_{1k} \qquad j = 1,2, \dots, n$$

$$RI_{rm_j}^{b} = \sum_{k=1}^{j-1} q_{rm_k} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3, \dots, n$$

$$RI_{rm_1}^{b} = 0$$

$$RI^{a}_{rm_j} = \sum_{k=1}^{j} \mathrm{q}_{\mathrm{rm_k}} - P_1 t_{P_1} \sum_{k=1}^{j-1} x_{1k} \qquad j = 2,3,\dots,n$$

$$RI^{a}_{rm_1} = q_{rm_1}$$

$$RH_v = \frac{h_v}{2} \sum_{i=1}^{2} \frac{d_i^2}{P_i\, n_i}$$

$$\mathrm{R}H_b = \frac{h_b}{2}\left[\sum_{j=2}^{n} \left(RI^{a}_{b_{j-1}} + RI^{b}_{b_j}\right) RT_j + \left(RI^{a}_{b_n} + RI^{b}_{b_1}\right)(\rho + RT_1)\right]$$

$$RH_r = \frac{h_r}{2}\left[\sum_{j=1}^{n} \left(RI_{r_{j-1}} + RI_{r_j}\right) RT_j + \left(RI_{r_n} + RI_{r_0}\right)\rho\right]$$

$$RH^{m}_{rm} = \frac{h_{rm}}{2}\left[\sum_{j=1}^{n-1} \left(RI^{a}_{rm_j} + RI^{b}_{rm_{j+1}}\right) RT_j + RI^{a}_{rm_n} RT_n\right] \qquad (100)$$

$$RI_{r_{j-1}} \geq (R - r) t_{P_2} x_{2j} \qquad j = 1,2,\dots,n$$

$$RI_{r_n} \geq 0$$

$$RI^{b}_{b_j} \geq 0 \qquad j = 1,2,\dots,n$$

$$RI^{a}_{rm_j} \geq P_1 t_{P_1} x_{1j} \qquad j = 1,2,\dots,n$$

$$RI^{b}_{rm_j} \geq 0 \qquad j = 1,2,\dots,n$$

$$RI^{a}_{rm_j} \geq 0 \qquad j = 1,2,\dots,n$$

$$q_{rm_j} \geq 0 \qquad j = 1,2,\dots,n$$

$$x_{ij}, y_{ij} \in \{0,1\} \qquad i = 1,2 \;\; and \;\; j = 1,2,\dots,n$$

$$a_j \in \{0,1\} \qquad j = 1,2,\dots,n$$

4.3. Solution Algorithm

It is clearly seen that the above three optimization models fall under the class of mixed-integer non-linear programming (MINLP) models, and that the cycle time (*T*) and the rest of the decision variables are separable in the objective functions which may be written in the form of $K = \frac{\text{A}}{T} + \text{B}T$, where A and B are functions of the problem parameters and the decision variables other that T. Moreover, for a given number of production batches (n_1, n_2), the production sequence, raw material ordering policy (a_j) and number of cycles to cover by one raw material shipment (u), and all the continuous variables, excluding the cycle time, can be determined using Equations (74) to (97). Recall from Chapter 3 that the cost function ($K = \frac{\text{A}}{T} + \text{B}T$) was proofed to be convex, in the cycle time. Hence, the same approach will be used to determine the optimal cycle time (T^*) that minimizes K.

4.3.1. **Optimal cycle time (T^*).** Utilizing the same approach followed in deriving the optimal cycle time in Chapter 3, the optimal cycle time (T^*), for a given number of production batches, is given by:

$$T^*_{OTO}\left(x_{ij}, a_j : i = 1,2 \; and \; j = 1,2, \dots, n\right) = \sqrt{\frac{O_v + nO_b + s_v}{(RH_{rm} + RH_r + RH_v + RH_b)}} \qquad (101)$$

$$T^*_{OTM}\left(u, x_{ij}, a_j : i = 1,2 \; and \; j = 1,2, \dots, n\right) = \sqrt{\frac{\frac{O_v}{u} + nO_b + s_v}{(RH^u_{rm} + RH_r + RH_v + RH_b)}} \qquad (102)$$

$$T^*_{MTO}\left(x_{ij}, a_j : i = 1,2 \; and \; j = 1,2, \dots, n\right) = \sqrt{\frac{O_v \sum_{j=1}^{n} a_j + nO_b + s_v}{(RH^m_{rm} + RH_r + RH_v + RH_b)}} \qquad (103)$$

We can clearly see that when $u = 1$ and $\sum_{j=1}^{n} a_j = 1$, Eq. (99) and Eq. (94) reduce to Eq. (73). This clearly indicates that 3S-CLSC-CS-OTO model is a special case of the other two models, namely 3S-CLCS-CS-OTM and 3S-CLCS-CS-MTO. Hence, dealing with the latter two models will guarantee the optimality of the decision made without explicitly considering 3S-FSC-CS-OTO model.

As shown earlier, in order to solve the models to optimality, we need to solve for the rest of the decision variables; namely $n, u, x_{ij} \; and \; a_j$. In order to find the optimal values of these variables $(n, u, x_{ij} \; and \; a_j)$, the following optimization problems are solved:

3S-CLSC-CS-OTM-T*

$$\text{Min}\ \left(\frac{O_v}{u} + nO_b + s_v\right)(RH_{rm}^u + RH_r + RH_v + RH_b)$$

s.t

All constraints of (3S-CLSC-CS-OTM) problem

3S-CLSC-CS-MTO-T*

$$\text{Min}\ \left(O_v \sum_{j=1}^{n} a_j + nO_b + s_v\right)(RH_{rm} + RH_r + RH_v + RH_b)$$

s.t

All constraints of (3S-CLSC-CS-MTO) problem

Despite the fact that the continuous variable (T) is no longer present in both problems, we can still see the existing nonlinearity in the objective function as well as in Constraints (82), (83), (97) and (100). Hence, the resulting problem, after solving for T^*, still falls under the class of MINLP models. Moreover, it would be intractable to attain closed form mathematical expressions to determine the optimal solution due to the difficulty of establishing the convexity of the non-differentiable objective function, especially for larger problem instances (i.e., large number of batches n). Alternatively, towards finding the optimal solution, an efficient iterative solution procedure that searches over a predetermined range for the values of the number of batches (n) is developed next.

4.3.2. **Solution procedure.** In the aforementioned analysis, it was assumed that production frequencies (n_1, n_2) are given. Now, an efficient solution procedure is proposed to solve the three-stage closed loop supply chain models under consignment stock partnership. First, it should be noted that this solution procedure proposed hereafter is built on the one proposed by Hariga et al. [24]. Furthermore, note that during one cycle, there are $(n-1)$ possible combinations for the number of production runs for the manufacturing and remanufacturing processes for a specific value of the total production runs (n), which are $\{(j, n-j): j = 1,2, \dots, n-1\}$.

In order to find the optimal solution of the three-stage closed loop supply chain models under consignment stock partnership, the iterative solution algorithm depicted in Figure 18 is used.

In this algorithm, the counter variable, v, is used as a stopping criterion due to the non-convexity of the objective function. The value of this counter is incremented by one every time the optimal total cost for a specific n is greater than that for $n-1$. When we realize 5 consecutive increases in the optimal total cost, we opt to terminate the search algorithm. As a result, we cannot claim the global optimality of the attained solution and whenever optimal is used, it refers to the local optimal solution.

After finding the optimal solution of 3S-CLSC-CS-OTM problem, the same algorithm is applied to the 3S-CLSC-CS-MTO problem and optimal total costs, under both models, are compared. The raw material ordering policy associated with the model yielding the lowest total cost is then considered to be the optimal policy minimizing the total cost per unit time. If the two models result in the same optimal total cost, this means that OTO raw material ordering policy is the optimal choice to be adopted. This is due to the fact that 3S-CLSC-CS-OTM and 3S-CLSC-CS-MTO models return the same solution if and only if both models reduce to the special case model, 3S-CLSC-CS-OTO.

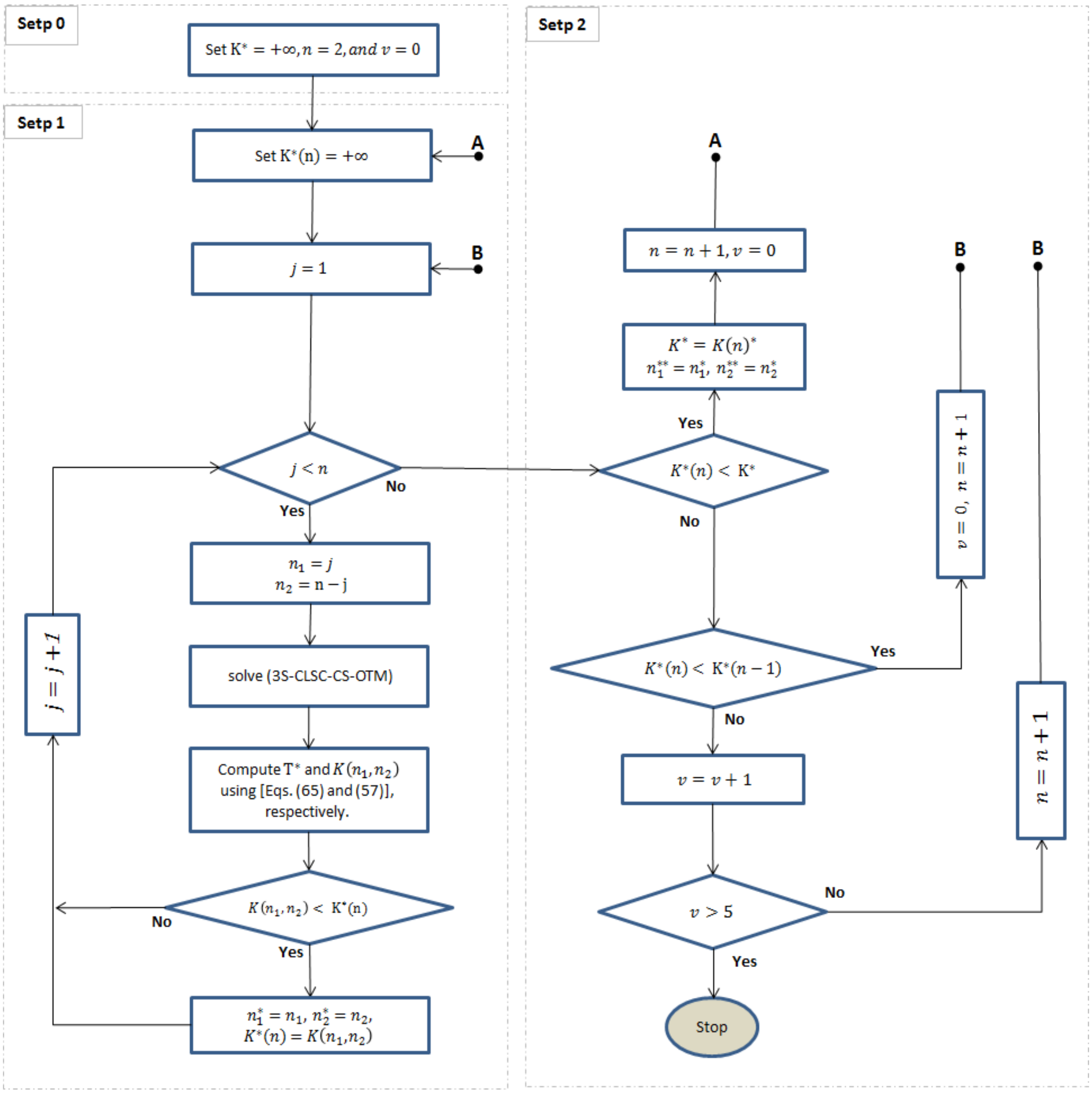


Figure 18: 3S-CLSC-CS solution algorithm

4.4. Computational Experiment

In this section, the proposed solution algorithm is first applied towards solving a numerical example to illustrate the impact of raw materials ordering policies and production sequence on the total system cost. A sensitivity analysis is then performed to provide more insights on the behavior of the models under different settings of the key problem parameters.

4.4.1. **Numerical example (base case).** The values of the input parameters to this example (base case) are: $S_1 = 200, S_2 = 250, O_b = 100, O_v = 200, D = 2000,\ d_1 = 1200, d_2 = 800,\ P_1 = 4000,\ R = 2000, h_v = 3, h_b = 4, h_{rm} = 2$, and $h_r = 1$. Note that these figures are mostly adopted from the relevant work of Hariga et al. [24] with the addition of raw material related cost parameters. The results obtained for different n values upon employing the proposed iterative solution algorithm are summarized in Tables 11 to 18 below. Note that the values given in boldface in the tables represent the optimal solution for that specific n, and the notions in the tables are defined as follow:

- *(R,M):* Consecutive production of remanufacturing batches followed by manufacturing batches.
- *(M,R):* Consecutive production of manufacturing batches followed by remanufacturing batches.
- j^{th} Q_{rm} *row*: The j^{th} production run at the beginning of which a raw material shipment order is placed.
- *RM policy row:* The optimal raw material ordering policy yielding the minimum total cost.

In addition to the above, note that the first newly manufactured batch takes place when the first raw material shipment order is placed. In other words, if a raw material order is going to be placed, it will take place only at the beginning of the manufacturing production run. Hence, the model synchronizes the raw material orders to take place only when needed for the production of new products. Notion 1 and 2 in the sequence define newly manufactured and remanufactured batches, respectively.

Table 11: Model's output for different total number of production runs (*n* = 2, 3 &4)

n	2	3		4		
n_1	**1**	1	**2**	1	**2**	3
n_2	**1**	2	**1**	3	**2**	1
T^*	**0.451**	0.524	**0.542**	0.549	**0.618**	0.606
$K(n_1, n_2)$	**3770**	3626	**3503**	3827	**3436**	3466
Sequence	**(R,M)**	(R,M)	**(M,R)**	(R,M)	**(R,M)**	(M,R)
J^{th} Q_{rm}	**2**	3	**1**	4	**3**	1
RM policy	**OTO**	OTO	**OTO**	OTO	**OTO**	OTO
$K^*(n)$	3769.9	3503.4		3435.5		

Table 12: Model's output for different total number of production runs (*n* = 5 & 6)

n	5				6				
n_1	1	**2**	3	4	1	2	**3**	4	5
n_2	4	**3**	2	1	5	4	**3**	2	1
T^*	0.573	**0.696**	0.643	0.633	0.597	0.723	**0.742**	0.674	0.658
$K(n_1, n_2)$	4013	**3427**	3576	3637	4188	3457	**3441**	3712	3797
Sequence	(R,M)	**(R,M)**	(R,M)	(M,R)	(R,M)	(R,M)	**(R,M)**	(M,R)	(M,R)
$J^{th}\ Q_{rm}$	5	**4**	3	1	6	5	**4**	1	1
RM policy	OTO	**OTO**	OTO	OTO	OTO	OTO	**OTO**	OTO	OTO
$K^*(n)$	3426.5				3441.4				

Table 13: Model's output for different total number of production runs (*n* = 7)

n	7					
n_1	1	2	3	**4**	5	6
n_2	6	5	4	**3**	2	1
T^*	0.620	0.750	0.818	**0.768**	0.715	0.684
$K(n_1, n_2)$	4356	3599	3483	**3409**	3779	3950
Sequence	(R,M)	(R,M)	(R,M)	**(R,M)**	(M,R)	(M,R)
$J^{th}\ Q_{rm}$	7	6	5	**4**	1	1
RM policy	OTO	OTO	OTO	**OTO**	OTO	OTO
$K^*(n)$	**3409.0**					

Table 14: Model's output for different total number of production runs (*n* = 8)

n	8						
n_1	1	2	**3**	4	5	6	7
n_2	7	6	**5**	4	3	2	1
T^*	0.642	0.777	**0.846**	0.844	0.795	0.984	0.708
$K(n_1, n_2)$	4517	3734	**3430**	3436	3650	3760	4097
Sequence	(R,M)	(R,M)	**(R,M)**	(R,M)	(R,M)	(11122111)	(M,R)
$J^{th}\ Q_{rm}$	8	7	**6**	5	4	1,6	1
RM policy	OTO	OTO	**OTO**	OTO	OTO	MTO	OTO
$K^*(n)$	3429.6						

Table 15: Model's output for different total number of production runs (*n* = 9)

n	9							
n_1	1	2	3	**4**	5	6	7	8
n_2	8	7	6	**5**	4	3	2	1
T^*	0.664	0.802	0.873	**0.907**	0.871	1.042	1.005	0.731
$K(n_1, n_2)$	4672	3863	3551	**3417**	3560	3744	3879	4239
Sequence	(R,M)	(R,M)	(R,M)	**(R,M)**	(R,M)	(11222 1111)	(11112 2111)	(M,R)
$J^{th}\ Q_{rm}$	9	8	7	**6**	5	1,6	1,7	1
RM policy	OTO	OTO	OTO	**OTO**	OTO	MTO	MTO	OTO
$K^*(n)$	3416.8							

Table 16: Model's output for different total number of production runs (*n* = 10)

n	10								
n_1	1	2	3	**4**	5	6	7	8	9
n_2	9	8	7	**6**	5	4	3	2	1
T^*	0.684	0.827	0.900	**0.945**	0.934	0.897	1.083	0.765	0.754
$K(n_1, n_2)$	4822	3988	3667	**3493**	3534	3678	3787	4312	4375
Sequence	(R,M)	(R,M)	(R,M)	**(R,M)**	(R,M)	(R,M)	(11222 11111)	(R,M)	(M,R)
$J^{th}\ Q_{rm}$	10	9	8	**7**	6	5	1,6	3	1
RM policy	OTO	OTO	OTO	**OTO**	OTO	OTO	MTO	OTO	OTO
$K^*(n)$	3492.7								

Table 17: Model's output for different total number of production runs (n = 11)

n	11									
n_1	1	2	3	4	**5**	6	7	8	9	10
n_2	10	9	8	7	**6**	5	4	3	2	1
T^*	0.70 5	0.85 2	0.926	0.972	**0.989**	0.960	0.923	0.870	0.846	0.777
$K(n_1, n_2)$	4967	4109	3779	3601	**3540**	3645	3792	4022	4138	4507
Sequence	(R,M)	(R,M)	(R,M)	(R,M)	**(R,M)**	(R,M)	(R,M)	(R,M)	(M,R)	(M,R)
$J^{th}\ Q_{rm}$	11	10	9	8	**7**	6	5	4	1	1
RM policy	OTO	OTO	OTO	OTO	**OTO**	OTO	OTO	OTO	OTO	OTO
$K^*(n)$	3540.4									

Table 18: Model's output for different total number of production runs (*n* = 12)

n	12										
n_1	1	2	3	4	**5**	6	7	8	9	10	11
n_2	11	10	9	8	**7**	6	5	4	3	2	1
T^*	0.724	0.875	0.951	0.998	**1.143**	1.126	0.986	1.163	1.159	1.098	0.798
$K(n_1, n_2)$	5108	4227	3888	3705	**3586**	3641	3752	3868	3881	4097	4635
Sequence	(R,M)	(R,M)	(R,M)	(R,M)	**(R,M)**	(R,M)	(R,M)	112222 111111	111222 111111	111111 221111	(M,R)
$J^{th}\ Q_{rm}$	12	11	10	9	**8,10**	7,10	6	1,7	1,7	1,9	1
RM policy	OTO	OTO	OTO	OTO	**MTO**	MTO	OTO	MTO	MTO	MTO	OTO
$K^*(n)$	3586										

Through examining the above tables, we can clearly notice the non-convexity of the total cost function with respect to the total number of batches n, where it can be seen that the total cost alternates between decreasing and increasing values for values of $n \leq 9$ followed by five consecutive increases starting from $n = 10$.

In addition, it can be observed that, for most values of n_1 and n_2, it is preferable to start the production with the remanufacturing process followed by the manufacturing process. However, when $n_1 = 2$ and $n_2 = 1$, the optimal sequence is to start by the manufacturing process followed by remanufacturing. For some combinations of n_1 and n_2 values, we can observe an intermittent production sequence where production starts by newly manufactured batches followed by a number of remanufactured batches and then followed by newly manufactured batches.

It is worth pointing that for larger values of n where $n \geq 8$, the MTO raw material ordering policy starts to be more cost effective than the OTO policy for some combinations of n_1 and n_2, especially when the cycle time gets large ($T > 1$), where in such situations two raw material orders are being placed instead of only one to cover the whole cycle production. Based on the output of the model, the optimal solution is to start with 3 consecutive remanufactured batches followed by 4 consecutive newly manufactured batches, with the OTO raw material ordering policy being the optimal one where the single raw material shipment is received at the beginning of the 4th production run. The optimal cycle time is found to be $T^* = 0.7681$, and the resulting chain-wide total cost is $3,409 per unit time.

4.4.2. **Sensitivity analysis.** This subsection provides the results of computational experiments of different problem instances under different settings of the problem parameters. In particular, we seek to study the impact of changing the cost parameters (holding, setup and ordering costs) on the behavior of the model. The results of such analysis shall illustrate the relevance and importance of the derived models and further provide more insights on the relationship between problem parameters and its impact on the decision variables.

In the following subsections, the column "RM policy" describes the optimal raw material ordering policy yielding the minimum total cost (i.e. One-To-One (OTO) policy, One-To-Multi (OTM) policy or Multi-To-One (MTO) policy).

4.4.2.1. ***Return rate impact.*** To assess the impact of the return rate (r) on the optimal solutions obtained, a one-way sensitivity analysis is conducted by varying this parameter while keeping all other problem parameters fixed as in the base case. Eight different values for r are chosen as follow $r =$ 400, 500, 600, 700, 800, 900, 1000 and 1100. These values are chosen in a way that assesses the impact of low and high return rates on the optimal solution obtained from the model. Hence, a total of 8 problem instances are analyzed for each of the two models with different ordering policies, and the optimal results are reported in Table 19 below.

Table 19: Effect of changing return rate value

r	n_1	n_2	T	Sequence	$j^{th}\ Q_{rm}$	RM Policy	TC
400	4	2	0.6706	(R,M)	3	OTO	3,809.2
500	3	2	0.6467	(R,M)	3	OTO	3,700.7
600	4	3	0.5656	(R,M)	4	OTO	3,605.0
700	4	3	0.5840	(R,M)	4	OTO	3,490.5
800	4	3	0.7681	(R,M)	4	OTO	3,409.0
900	4	4	0.8587	(R,M)	5	OTO	3,303.0
1000	3	3	0.7524	(R,M)	4	OTO	3,192.4
1100	3	4	0.8553	(R,M)	5	OTO	3,068.2

It is clearly noted that, for both models, increasing the return rate results in increasing the number of remanufactured batches (n_2) without having a consistent impact on the number of newly manufactured batches (n_1). This makes sense as a bigger portion of the demand is now being satisfied from returned products. Whenever there is an increase in the value of n_2 comes also an increase in the cycle time T to allow for the production of more batches.

It is worth mentioning that both models start the production by the remanufacturing process first followed by the manufacturing process (R, M) for all problem instances. This indicates that the production sequence is not being affected by the value of the return rate when comparing the MTO to the OTM three-stage closed-loop supply chain systems.

In addition to that, results show that both models, 3S-CLSC-CS-OTM and 3S-CLSC-CS-MTO, are performing the same for the different values of r and that the optimal raw material ordering policy for all instances is the OTO policy.

4.4.2.2. ***Impact of buyer's ordering and holding costs.*** To assess the impact of the buyer's ordering and holding costs on the optimal solutions obtained, those cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. As the ordering cost typically increases in the upstream direction, five different values for O_b are chosen through multiplying its base value of $O_b = 100$ by 0.05, 0.10, 1, 1.5 and 2. Those multipliers are chosen to cover a wide range and at the same time to be consistent with the ordering cost of the vendor ($O_v \geq O_b$) and the holding cost of the buyer ($O_b \geq h_b$). Furthermore, as the holding cost typically increases in the downstream direction, five different values for h_b are chosen for each of the abovementioned O_b values; $h_b = 3\ (= h_v), 3.5, 4, 4.5\ and\ 5$. Hence, a total of 25 problem instances are analyzed for each of the two models with different ordering policies, and the optimal results are reported in Table 20.

It is clearly seen that, for both models, as the value of the buyer's ordering cost increases, it becomes more economical to have fewer orders of larger sizes as the value of n decreases but with no major impact on the cycle time T. As expected, however, altering the ordering cost of the buyer has no effect on the number of raw material orders and accordingly the optimal raw material ordering policy for all tested instances is the OTO policy. For the buyer's holding cost, altering its value has no impact on the total number of production batches or the production sequence for the same buyer ordering cost. However, T gets smaller as h_b is getting higher in order to ensure that smaller batches are being ordered which minimizes the holding cost at the buyer's end.

Table 20: Effect of changing buyer's ordering and holding costs

O_b	h_b	n_1	n_2	T	Sequence	$j^{th}\ Q_{rm}$	RM policy	TC
5	3	11	13	0.762	(R,M)	14	OTO	2,018.1
5	3.5	11	13	0.734	(R,M)	14	OTO	2,097.3
5	4	11	13	0.708	(R,M)	14	OTO	2,173.7
5	4.5	11	13	0.681	(R,M)	14	OTO	2,250.9
5	5	11	13	0.663	(R,M)	14	OTO	2,318.8
10	3	8	9	0.764	(R,M)	10	OTO	2,144.3
10	3.5	8	9	0.734	(R,M)	10	OTO	2,229.2
10	4	8	9	0.708	(R,M)	10	OTO	2,311.0
10	4.5	8	9	0.685	(R,M)	10	OTO	2,389.9
10	5	8	9	0.663	(R,M)	10	OTO	2,466.3
100	3	4	3	0.834	(R,M)	4	OTO	3,151.0
100	3.5	4	3	0.799	(R,M)	4	OTO	3,282.5
100	**4**	4	3	0.768	(R,M)	4	OTO	3,409.0
100	4.5	4	3	0.740	(R,M)	4	OTO	3,530.9
100	5	4	3	0.716	(R,M)	4	OTO	3,648.8
150	3	5	1	0.794	(M,R)	1	OTO	3,429.4
150	3.5	5	1	0.762	(M,R)	1	OTO	3,578.1
150	4	5	1	0.733	(M,R)	1	OTO	3,720.9
150	4.5	5	1	0.708	(M,R)	1	OTO	3,858.4
150	5	5	1	0.685	(M,R)	1	OTO	3,991.1
200	3	4	1	0.814	(M,R)	1	OTO	3,597.5
200	3.5	4	1	0.786	(M,R)	1	OTO	3,719.5
200	4	4	1	0.758	(M,R)	1	OTO	3,863.8
200	4.5	4	1	0.732	(M,R)	1	OTO	4,003.0
200	5	4	1	0.708	(M,R)	1	OTO	4,137.4

It is also worth mentioning that as the value of the buyer's ordering cost gets closer to that of the vendor (=200), $O_b \geq 150$, it would be preferable to start by producing new batches first followed by remanufactured batches. However, when O_b gets smaller (<150) it would be preferable to start by producing remanufactured batches followed by newly manufactured batches. As mentioned earlier in the case of (M, R) production sequence, we only have one remanufacturing batch that follows a number of newly manufactured batches. Based on the optimal solutions obtained from the different values of O_b and h_b, the sequence (M, R) turns out to be the optimal one in 40% of all problem instances tested (10 out of 25 instances).

Finally, both models, 3S-CLSC-CS-OTM and 3S-CLSC-CS-MTO, yield the same results for the different values of O_b and h_b and that the optimal raw material ordering policy for all instances is the OTO policy. Moreover, we always order raw materials just in time for the production of the first newly manufactured batch.

4.4.2.3. ***Impact of vendor's holding cost and manufacturing setup cost.*** To gain better insights on the impact of the vendor's holding cost and the manufacturing setup cost on the optimal solutions obtained, these cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. To cover a wide range of values, five different values for S_1 are chosen through multiplying its base value of $S_1 = 200$ by 0.05, 0.10, 1, 5 and 10. Since the holding cost typically increases in the downstream direction ($h_{rm} \leq h_v \leq h_b$), five different values for h_v are chosen for each of the aforementioned S_1 values; $h_v = 2(= h_{rm}), 2.5, 3, 3.5\ and\ 4(= h_b)$. Hence, a total of 25 problem instances are analyzed for each of the two models with different ordering policies, and the optimal results are reported in Table 21.

It can be seen that the total number of production batches (n) stays the same with increasing values for the vendor's holding cost (when $S_1 = 10\ and\ 20$) or slightly increases for the higher values of S_1. In either case, increasing the vendor's holding cost results in a reduction in the cycle time (T) indicating that the model prefers the production of smaller sizes batches, to reduce the inventory levels and accordingly the inventory holding cost at the vendor's premises. As anticipated, altering the vendor's holding cost had no evident effect on the raw materials replenishment policy. However, it is clearly seen that the manufacturing setup cost, S_1, has an impact on the decision variables n and T as well as the raw material replenishment policy along with the production sequence. Since the setup cost is incurred only once at the beginning of the manufacturing process, increasing the setup cost results in increasing the cycle time to achieve economies of scale and overcome the high setup cost through the production of larger sized batches.

Table 21: Effect of changing vendor's holding cost and manufacturing setup cost

S_1	h_v	n_1	n_2	T	Sequence	$j^{th}\ Q_{rm}$	RM policy	TC
10	2	4	1	0.600	(M,R)	1	OTO	2,856.1
10	2.5	4	1	0.588	(M,R)	1	OTO	2,908.4
10	3	4	1	0.578	(M,R)	1	OTO	2,959.7
10	3.5	4	1	0.568	(M,R)	1	OTO	3,010.1
10	4	4	1	0.558	(M,R)	1	OTO	3,059.7
20	2	4	1	0.603	(M,R)	1	OTO	2,866.8
20	2.5	4	1	0.591	(M,R)	1	OTO	2,919.2
20	3	4	1	0.581	(M,R)	1	OTO	2,970.7
20	3.5	4	1	0.571	(M,R)	1	OTO	3,021.4
20	4	4	1	0.561	(M,R)	1	OTO	3,071.1
200	2	3	1	0.631	(M,R)	1	OTO	3,310.8
200	2.5	3	3	0.751	(R,M)	4	OTO	3,369.4
200	**3**	4	3	0.768	(R,M)	4	OTO	3,409.0
200	3.5	4	3	0.760	(R,M)	4	OTO	3,448.1
200	4	4	3	0.752	(R,M)	4	OTO	3,486.7
1000	2	5	5	1.267	(R,M)	6,8	MTO	4,148.8
1000	2.5	5	5	1.254	(R,M)	6,8	MTO	4,193.2
1000	3	5	5	1.242	(R,M)	6,8	MTO	4,237.1
1000	3.5	5	5	1.229	(R,M)	6,8	MTO	4,280.6
1000	4	5	5	1.218	(R,M)	6,8	MTO	4,323.6
2000	2	6	6	1.571	(R,M)	7,10	MTO	4,861.4
2000	2.5	6	6	1.557	(R,M)	7,10	MTO	4,907.0
2000	3	6	6	1.543	(R,M)	7,10	MTO	4,952.1
2000	3.5	6	6	1.530	(R,M)	7,10	MTO	4,996.8
2000	4	6	6	1.517	(R,M)	7,10	MTO	5,041.1

In addition, note that for high values of the setup cost, namely $S_1 = 1000\ and\ 2000$, there is also a noticeable increase in the number of production batches (n) coupled with the previously noted increase in the cycle time, which would both serve the purpose of minimizing the setup cost per unit time, as less frequent setups would be needed. For these large values of the setup cost ($S_1 = 1000\ and\ 2000$), 3S-CLCS-CS-MTO model increases the number of raw material shipments per cycle, following the sharp increase in the values of n and T, as to minimize the inventory holding cost of raw materials. This would naturally result in lower raw material holding cost on the expense of a higher ordering cost of these material. However, the attained tradeoff between ordering and holding costs will lead to the minimum chain wide total cost. As a result, we can see that the optimal solutions for $S_1 \geq 1000$ is gained by adopting the MTO raw material ordering policy where two orders are placed per cycle. The first order of raw material is always placed just in time for the start of the newly manufacturing batches. For lower values of S_1, however, the OTO raw material ordering policy is the dominant policy that will guarantee the minimum integrated total cost.

It is worth mentioning that when the manufacturing setup cost of the vendor gets lower (< 200), it would be preferable to start by producing new batches followed by remanufactured batches. However, when S_1 gets larger (≥ 200) it would be preferable to start by producing remanufactured batches first followed by newly manufactured batches. As mentioned earlier, when (M, R) production sequence is adopted, we only have one remanufacturing batch and the remaining batches are being produced through the newly manufacturing process. Based on the optimal solutions obtained under different values of S_1 and h_v, the sequence (M, R) gained a weight of 44% in all problem instances solved.

4.4.2.4. ***Impact of returned products holding cost and remanufacturing setup cost.*** In order to also gain better insights on the impact of the vendor's returned products holding cost as well as the remanufacturing setup cost on the optimal solutions obtained, again those cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. To that end, five different values for S_2 are chosen through multiplying its base value of $S_2 = 250$ again by 0.05, 0.10, 1, 5 and 10. In order for remanufacturing to be an economically viable option, the purchasing cost, and accordingly the holding cost, of the returned products is assumed to be cheaper than that of buying new raw material (i.e. $h_r \leq h_{rm}$). As such, five different values for h_r are chosen for each of the aforementioned S_2 values; where $h_r = 0.25, 0.5, 1, 1.5 \; and \; 2 (= h_{rm})$. Hence, a total of 25 problem instances are analyzed for each of the two models with different ordering policies, and the optimal results are reported in Table 22.

It can be noted that the total number of production batches (n) decreases as h_r value increases for high setup cost values ($S_2 \geq 250$) or remains as is when for smaller setup costs ($S_2 < 250$) for increasing h_r values. In either case, when the cost of holding the returned products increases, this results in a decrease in the cycle time (T) implying that smaller batches are being produced. As the value of h_r becomes equal to that of h_{rm}, this may cause the model to alter the production sequence from (R, M) to (M, R), as can be seen for the case where $S_2 = 250\ and\ 1250$, where only one remanufactured batch is then produced. Hence, altering the value of h_r may have an impact on the resulting production sequence.

Table 22: Effect of changing buyer's holding cost and remanufacturing setup cost

S_2	h_r	n_1	n_2	T	Sequence	$j^{th}\ Q_{rm}$	RM policy	TC
12.5	0.25	3	1	0.551	(M,R)	1	OTO	2,920.7
12.5	0.5	3	1	0.545	(M,R)	1	OTO	2,953.9
12.5	1	3	1	0.533	(M,R)	1	OTO	3,019.2
12.5	1.5	3	1	0.522	(M,R)	1	OTO	3,083.1
12.5	2	3	1	0.512	(M,R)	1	OTO	3,145.7
25	0.25	3	1	0.555	(M,R)	1	OTO	2,943.1
25	0.5	3	1	0.549	(M,R)	1	OTO	2,976.5
25	1	3	1	0.537	(M,R)	1	OTO	3,042.3
25	1.5	3	1	0.526	(M,R)	1	OTO	3,106.7
25	2	3	1	0.516	(M,R)	1	OTO	3,169.8
250	0.25	4	3	0.800	(R,M)	4	OTO	3,263.2
250	0.5	4	3	0.789	(R,M)	4	OTO	3,312.5
250	**1**	4	3	0.768	(R,M)	4	OTO	3,409.0
250	1.5	3	3	0.723	(R,M)	4	OTO	3,500.9
250	2	3	1	0.582	(M,R)	1	OTO	3,576.0
1250	0.25	5	5	1.361	(R,M)	6,9	MTO	4,154.0
1250	0.5	5	5	1.335	(R,M)	6,8	MTO	4,235.5
1250	1	5	5	1.288	(R,M)	6,9	MTO	4,394.1
1250	1.5	5	5	1.245	(R,M)	6,8	MTO	4,547.1
1250	2	4	1	0.831	(M,R)	1	OTO	4,665.0
2500	0.25	6	7	1.796	(R,M)	8,11	MTO	4,924.2
2500	0.5	7	7	1.773	(R,M)	8,12	MTO	5,021.7
2500	1	7	7	1.703	(R,M)	8,12	MTO	5,232.3
2500	1.5	6	6	1.574	(R,M)	7,10	MTO	5,427.1
2500	2	6	6	1.522	(R,M)	7,10	MTO	5,614.1

Similar to the reported impact of S_1 on the decision variables n and T as well as the raw material replenishment policy and production sequence, increasing the setup cost (S_2) results in increasing the cycle time to overcome the high setup cost which will in turn result in lower setup cost per unit time for larger T compared with smaller T. Similar to the new manufacturing process, increasing the setup cost increases the number of batches shipped to the buyer (n) such that less frequent setups are needed per unit of time.

Moreover, the resulting raw material replenishment policy and production sequence exhibited a same response pattern to that observed in the case of changing S_1 and h_v values. In an attempt to minimize the inventory holding cost of raw materials, 3S-CLCS-CS-MTO model increases the number of raw material shipments per cycle for values of the setup cost ≥ 1250. This will clearly result in lower holding cost but higher ordering cost of raw material. As a result of optimizing the raw material ordering policies, we can see that the optimal solutions for $S_2 \geq 1250$ is gained by adopting the MTO raw material ordering policy, rather than the OTM or OTO raw material ordering policies, where now two orders per one cycle are placed. In addition, in an attempt of getting the lowest possible raw material holding cost, the first order of raw material is always ordered just in time for the start of the newly manufactured batches. For lower values of S_2, 3S-CLCS-CS-OTM and 3S-CLCS-CS-MTO models perform the same and reduce to 3S-CLCS-CS-OTO model.

Considering the production sequence, it is worth pointing out that when the setup cost of remanufacturing gets lower (< 250), it would be preferable to start by producing new batches first followed by remanufactured batches. However, when S_2 gets larger (≥ 250) it would be preferable to start by producing remanufactured batches followed by newly manufactured batches. As mention earlier, when (M, R) production sequence is adopted, we only have one remanufactured batch and the remaining batches are being produced through the newly manufacturing batches. Based on the optimal solutions obtained, the sequence (M, R) tuned out to the optimal in 48% of the 25 tested problem instances.

4.4.2.5. ***Impact of manufacturing and remanufacturing setup costs.*** To assess the impact of changing the setup costs together, on the optimal solutions obtained, again those cost parameters are altered, one at a time, while keeping all other problem parameters fixed as in the base case. To cover a wide range of the setup costs, five different values for S_1 and S_2 are chosen through multiplying their base values of 200

and 250, respectively, by 0.05, 0.10, 1, 5 and 10. Hence, a total of 25 problem instances are analyzed for each of the two models with different ordering policies, and the optimal results are reported in Table 23.

Table 23: Effect of changing manufacturing and remanufacturing setup costs

S_1	S_2	n_1	n_2	u	T	Sequence	$j^{th}\ Q_{rm}$	RM policy	TC
10	12.5	1	1	2	0.245	(R,M)	2	OTM	2,634.4
10	25	3	1	-	0.471	(M,R)	1	OTO	2,669.1
10	250	4	1	-	0.578	(M,R)	1	OTO	2,959.7
10	1250	9	3	-	1.310	(111222111111)	1,7	MTO	4,238.1
10	2500	6	3	-	1.458	(112221111)	1,6	MTO	5,093.4
20	12.5	3	1	-	0.470	(M,R)	1	OTO	2,663.8
20	25	3	1	-	0.475	(M,R)	1	OTO	2,690.0
20	250	4	1	-	0.581	(M,R)	1	OTO	2,970.7
20	1250	4	4	-	1.056	(R,M)	5	OTO	4,294.9
20	2500	6	3	-	1.462	(112221111)	1,6	MTO	5,107.0
200	12.5	3	1	-	0.533	(M,R)	1	OTO	3,019.2
200	25	3	1	-	0.537	(M,R)	1	OTO	3,042.3
200	**250**	4	3	-	0.768	(R,M)	4	OTO	3,409.0
200	1250	5	5	-	1.288	(R,M)	6,9	MTO	4,394.1
200	2500	7	7	-	1.703	(R,M)	8,12	MTO	5,232.3
1000	12.5	4	4	-	0.994	(R,M)	5	OTO	4,044.0
1000	25	4	4	-	0.998	(R,M)	5	OTO	4,056.5
1000	250	5	5	-	1.242	(R,M)	6,9	MTO	4,237.1
1000	1250	6	6	-	1.543	(R,M)	7,10	MTO	4,952.1
1000	2500	7	7	-	1.919	(R,M)	8,11,13	MTO	5,670.5
2000	12.5	6	6	-	1.495	(R,M)	7,10	MTO	4,796.9
2000	25	6	6	-	1.497	(R,M)	7,10	MTO	4,805.2
2000	250	6	6	-	1.543	(R,M)	7,10	MTO	4,952.1
2000	1250	7	7	-	1.804	(R,M)	8,12	MTO	5,542.9
2000	2500	7	7	-	2.086	(R,M)	8,11,13	MTO	6,164.5

It can be clearly noted that the total number of production batches (n) increases with increasing either value of the setup costs. In addition, the cycle time (T), gets larger when increasing the manufacturing setup cost (S_1) and keeping S_2 the same, and vice versa. However, the lot sizing policy does not seem to follow a particular trend with changing values for the setup cost even though both, n and T, assume increasing values. This is due to the fact that the lot size ($Q = \frac{DT}{n}$) also depends on the increase in the magnitude of both variables since one is having a direct proportional effect (T) while the other has a reverse proportional impact (n).

Moreover, there is a clear impact on the resulting raw material replenishment policy as a result to changing S_1 and S_2 values. In an attempt to minimize the inventory holding cost of raw materials, 3S-CLCS-CS-MTO model increases the number of raw material shipments per cycle for values of $S_2 \geq 1250$ when $S_1 \leq 250$, values of $S_2 \geq 250$ when $S_1 = 1000$, and for all values of S_2 when $S_1 = 2000$. This clearly validates that for larger values of S_2, it turned out to be economically advantageous to have multiple raw material orders per one cycle due to the large increase in the cycle time, yielding lower holding cost of raw materials. In contrast, 3S-CLCS-CS-OTM model increases the number of cycles to be covered by one raw material shipment for the smallest values of $S_1 (= 12)$ and $S_2 (= 12.5)$ in an attempt to minimize the ordering cost of raw material per cycle. This clearly show that for small values of S_1 and S_2 and when the cycle time is very small, it turned out to be economically advantageous to have one raw material order to cover multiple cycles, yielding lower ordering cost of raw materials. In addition, towards getting the lowest possible raw material holding cost, the first order of raw material is always placed just in time for the start of the production of newly manufactured batches.

In addition, the production sequence also has been impacted by varying the setup costs. Once can clearly see that when the setup costs are greater than or equal the base values ($S_1 \geq 200, S_2 \geq 250$), it would be preferable to start by producing remanufactured batches followed by newly manufactured ones. However, when the setup costs are less than the base values ($S_1 < 200, S_2 < 250$), it would be preferable to start by producing new batches first followed by remanufactured ones, except for the case when the OTM raw material ordering policy is adopted. It is worth pointing out that some instances showed intermittent sequences as the optimal sequence in case of high setup costs for the remanufactured batches and low setup costs for the newly manufactured ones. When $S_1 = 10 \; and \; 20, \; S_2 = 2500$), it would be preferable to start by producing two consecutive new batches first followed by three consecutive remanufactured batches, and then followed by four consecutive new batches. In case when $S_1 = 10$, $S_2 = 1250$), it would be preferable to start by producing three consecutive new batches first followed by three consecutive remanufactured batches, and then followed by six consecutive new batches. Based on the optimal solutions obtained, the sequence (M, R) turned out to be the optimal sequence in 28% of the 25 tested problem instances.

4.4.2.6. ***Impact of vendor's ordering and holding costs of raw materials.*** It is particularly important to analyze the impact of the vendor's ordering and holding costs of raw materials on the optimal values of the decision variables and hence optimal solutions obtained. To that sake, those cost parameters are altered, one at a time, while keeping all other problem parameters fixed at their base value. As the ordering cost typically increases in the upstream direction, five different values for O_v are chosen: $O_v = O_b (= 100)$, 200, 400, 800 and 2000. On the other hand, since the holding cost typically increases in the downstream direction ($h_r \leq h_{rm} \leq h_v$), five different values for h_{rm} are chosen for each of the aforementioned O_v values; namely $h_{rm} = 1 (= h_r), 1.5, 2, 2.5 \; and \; 3 (= h_v)$. Hence, a total of 25 problem instances will are for each of the two models with different ordering policies, and the optimal results are reported in Table 24.

One can clearly note that changing the cost parameters related to raw materials have a significant impact on the values of the model's decision variables. In particular, increasing the ordering cost of raw materials is highly impacting the cycle time (T), where it typically results in increasing the cycle time so that less frequent orders are made. This, coupled with an increase in the total number of batches (n) ensures that the ordering cost at the vendor's end is minimized. However, increasing the holding cost of raw materials (h_{rm}) results in decreasing the cycle time (T) with some exceptions. It is also clearly seen that altering the value of h_{rm} has a negligible impact on the total number of production batches (n) when $O_v \leq 400$ and no impact for larger values of $O_v (> 400)$ with the case of $O_v = 2000$ being an exception.

In addition to the previously mentioned points, we can note that 3S-FSC-CS-OTM model reduces the number of cycles to be covered by one raw material shipment (u) as h_{rm} is getting higher $(O_v = 2000)$, while it calls for covering two cycles in one raw material shipment $(u = 2)$ for low holding cost value. This is obviously happening to minimize the inventory levels as its holding cost is getting higher. However, 3S-FSC-CS-MTO gets benefited from its multiple orders when the holding cost is relatively low compared to the ordering cost.

Table 24: Effect of changing vendor's ordering and holding costs of raw materials

O_v	h_{rm}	n_1	n_2	u	T	Sequence	j^{th} Q_{rm}	RM policy	TC
100	1	5	1	-	0.652	(M,R)	1	OTO	3,115.7
100	1.5	5	1	-	0.642	(M,R)	1	OTO	3,159.9
100	2	3	1	-	0.576	(M,R)	1	OTO	3,264.7
100	2.5	4	3	-	0.792	(R,M)	4,6	MTO	3,310.8
100	3	4	3	-	0.784	(R,M)	4,6	MTO	3,349.4
200	1	4	3	-	0.800	(R,M)	4	OTO	3,263.2
200	1.5	4	3	-	0.784	(R,M)	4	OTO	3,336.9
200	**2**	4	3	-	0.768	(R,M)	4	OTO	3,409.0
200	2.5	3	3	-	0.727	(R,M)	3	OTO	3,479.5
200	3	3	1	-	0.588	(M,R)	1	OTO	3,540.6
400	1	4	4	-	0.943	(R,M)	5	OTO	3,495.7
400	1.5	4	3	-	0.840	(R,M)	4	OTO	3,575.5
400	2	4	3	-	0.823	(R,M)	4	OTO	3,652.8
400	2.5	4	3	-	0.807	(R,M)	4	OTO	3,728.5
400	3	4	3	-	0.792	(R,M)	4	OTO	3,802.6
800	1	4	4	-	1.051	(R,M)	5	OTO	3,896.4
800	1.5	4	4	-	1.027	(R,M)	5	OTO	3,990.0
800	2	4	4	-	1.004	(R,M)	5	OTO	4,081.5
800	2.5	4	4	-	0.982	(R,M)	5	OTO	4,170.9
800	3	4	4	-	0.962	(R,M)	5	OTO	4,258.4
2000	1	4	4	2	0.957	(R,M)	5	OTM	4,697.2
2000	1.5	6	6	-	1.462	(R,M)	7	OTO	4,842.2
2000	2	6	6	-	1.425	(R,M)	7	OTO	5,081.8
2000	2.5	6	6	-	1.391	(R,M)	7	OTO	5,194.5
2000	3	6	6	-	1.359	(R,M)	7	OTO	5,296.4

Moreover, altering O_v and h_{rm} affects the raw material replenishment policy as well as the production sequence. Note that both models perform the same for all problem instances except for those when the MTO raw material ordering policy is more cost effective due to its flexibility in reducing the holding cost of raw materials through having multiple orders ($O_v = 100, h_{rm} = 2.5\ and\ 3$), and when the OTM raw material ordering policy is more cost effective due to its flexibility in reducing the ordering cost of raw materials by having one order that covers two cycles ($O_v = 2000, h_{rm} = 1$). This is obviously happening to reduce the ordering/holding cost of raw materials per cycle which is the main advantage of considering such ordering policies of raw materials at the first place.

The minimum total cost will be obtained by optimizing the tradeoff between ordering and holding costs. As a result of optimizing the raw material ordering policies, we can see how 3S-CLSC-CS-OTM and 3S-CLSC-CS-MTO models have been optimized when $O_v = 2000 \, and \, 100$, respectively.

Considering the production sequence, it is worth noting that when $O_v \leq 100$, it would be preferable to start by producing new batches followed by remanufactured batches. However, when $O_v > 100$, it would be preferable to start by producing remanufactured batches followed by newly manufactured batches. As already mentioned earlier, when (M, R) production sequence is adopted, we only have one remanufactured process and the remaining batches are being produced through the new manufacturing processes. Based on the optimal solutions obtained under different values of O_v and h_{rm}, the sequence (M, R) gained a lesser weight of 16% in all problem instances compared with other parameters studied earlier.

4.5. Conclusion

This chapter illustrated the impact of incorporating raw material related costs (ordering and holding) on the production sequence and other decision variables as well as the chain wide total cost of a centralized closed-loop supply chain composed of a single supplier, a single vendor/manufacturer and a single buyer operating with CS partnership. Three mathematical models have been developed according to the three raw material replenishment policies used.

Mathematical models and results showed that 3S-CLSC-CS-OTO model is a special case of the other two models. In addition to that, results validated that raw material cost components do have a major impact on the related decision variables and hence on the long-run system wide cost when operating under manufacturing-remanufacturing policy.

Among all the 133 problem instances that were analysed in the sensitivity analysis section, (M, R) sequence turned out to be the optimal production sequence in 33% of all sequences that resulted in the optimal solution while (R, M) and intermitted sequences turned out to be the optimal production sequence in 65% and 2%, respectively, of all sequences that resulted in the optimal solution.

To validate the value of the work developed throughout this thesis, tests and comparisons with the existing work are presented in the following chapter.

Chapter 5: Models Validation and Comparisons

This chapter presents a comparison of the models developed in the previous two chapters under different settings of the problem parameters, in order to assess the economic benefits of the remanufacturing of returned items. In addition, a thorough comparison of the three-stage CLSC models presented in this thesis with the two-stage CLSC model of Hariga et al. [24] is also conducted. The purpose is to show the impact of adding one more stage to the supply chain system under study on the optimal solutions found which in turn validates the relevance and importance of the work presented in this thesis.

First and foremost, it is important to point out that Hariga et al. [24] showed that it is more economical to have a CLSC than a FSC, or what the authors call "manufacturing only policy", for a two-stage supply chain system comprised of a single vendor and a single buyer. However, to the best of our knowledge, there is no previous work that has established a similar comparison in the context of a three-stage supply chain system with no prior restrictions on the production sequence. Hence, in this chapter, we will be comparing the 3S-FSC-CS, developed in Chapter 3, with the 3S-CLSC-CS systems of Chapter 4, in order to assess the economic benefits of collecting the returned products and remanufacturing them.

In addition, for the two-stage supply chain system, Hariga et al. [24] found that (R,M) and (M,R) production sequences turned out to be the optimal sequences in 87% and 10% of all the reported problem instances, respectively. However, the effect of adding one more stage on the production sequence of a closed-loop system has not yet been studied in the literature. Hence, this chapter also assesses the impact of incorporating the raw material replenishment policy on mainly the production sequence, along with other decision variables, through comparing Hariga et al. [24] model, hereafter referred to as 2S-CLSC-CS model, with the 3S-CLSC-CS developed in Chapter 4.

5.1. Comparing the Forward with Closed-Loop Supply Chain of a Three-Stage System

This section illustrates the impact of having a remanufacturing process in a three-stage supply chain system with no prior restrictions on the production sequence. The same set of values for the problem parameters that were used in Chapter 4 will be used here to compare both models, namely $S_1 = 200, S_2 = 250, O_b = 100, O_v = 200, D = 2000,\ d_1 = 1200, d_2 = 800, P_1 = 4000,\ R = 2000,\ h_b = 4,\ h_v = 3, h_{rm} = 2$ and $h_r = 1$. The optimal solution for each instance will be reported and then the percentage of cost deviation between the FSC and CLSC will be calculated, after incorporating the material cost ($TC + MC$). Note that positive percentage of cost deviation means savings in the total cost when having a CLSC system over a "manufacturing only policy", and that the unit cost of materials for the newly manufactured products (C_1) and remanufactured products (C_2) are assumed to be 10 and 5, respectively. Accordingly, the material cost of the CLSC system for each problem instance is calculated as follow: $MC = C_1 d_1 + C_2 d_2$, and the material cost of the FSC system for each instance is simply: $MC = C_1 D$. Note that TC is the optimal chain wide total cost obtained by solving the models.

5.1.1. **Return rate impact.** To assess the impact of the remanufacturing process on the three-stage system when changing the remanufacturing rate (r), recall the optimal solutions for the different values of (r) which are reported in Table 25. Note that the 3S-FCS-CS model has only one solution regardless of the value of (r), which is $n = 2, T\ =\ 0.355,\ TC = 3{,}376\ and\ TC\ +\ MC\ =$ \$23,376.

Table 25: Percentage cost deviation for different return rate values

	3S-CLSC-CS					% Cost deviation
r	n_1	n_2	T	TC	$TC + MC$	
400	4	2	0.6706	3,809.2	21,809	**7%**
500	3	2	0.6467	3,700.7	21,201	**10%**
600	4	3	0.5656	3,605.0	20,605	**13%**
700	4	3	0.5840	3,490.5	19,990	**17%**
800	4	3	0.7681	3,409.0	19,409	**20%**
900	4	4	0.8587	3,303.0	18,803	**24%**
1000	3	3	0.7524	3,192.4	18,192	**28%**
1100	3	4	0.8553	3,068.2	17,568	**33%**

The results reported clearly show that for all values of (r), it is economically beneficial to have the remanufacturing process in the system than the manufacturing only option. In addition to that, the higher the remanufacturing rate the more savings are generated from the remanufacturing process. This actually is due to the savings obtained in the material cost when operating under CLSC system as compared to the FCS.

5.1.2. **Remanufacturing setup cost impact.** In order to also assess the impact of the remanufacturing setup cost (S_2) on the percentage cost deviation, five different values of S_2 are chosen while keeping all other problem parameters at their base values. Note that the 3S-FCS-CS model has only one solution regardless of the value of (S_2), where this solution is $n = 2, T = 0.355,\ TC = 3{,}376\ and\ TC + MC = \$23{,}376$.

Table 26: Percentage cost deviation for different Remanufacturing setup cost values

	3S-CLSC-CS					% Cost deviation
S_2	n_1	n_2	T	TC	$TC + MC$	
12.5	3	1	0.533	3,019	19,019	**23%**
25	3	1	0.537	3,042	19,042	**23%**
250	4	3	0.768	3,409	19,409	**20%**
1250	5	5	1.288	4,394	20,394	**15%**
2500	7	7	1.703	5,232	21,232	**10%**

It is clearly shown in Table 26 that for all values of S_2, it is more economical to have the remanufacturing process in the system than manufacturing only. However, the higher the remanufacturing setup cost the less savings are realized from the remanufacturing process. This reduction in the savings is due to the increased TC for the CLSC compared to the fixed base TC of FCS for increasing values of S_2.

5.1.3. **Vendor's holding cost of finished products and manufacturing setup cost impact.** We also analyze the impact of varying the vendor's holding cost of the finished products as well as the setup cost for the newly manufactured products on the cost savings obtained. The results in Table 27 show that on average we are saving 23% of the cost when operating with manufacturing-remanufacturing policy once compared to the "manufacturing-only policy".

Moreover, it can be seen that the holding cost of finished products has a negligible impact on the percentage of saving as well as the lower values of manufacturing setup cost. However, a high increase in the manufacturing setup cost results in more savings. This behavior is due to a higher increase in TC and MC in the FCS system as compared to that of the CLSC.

Table 27: Percentage cost deviation for different vendor's holding cost and manufacturing setup cost values

		3S-CLSC-CS					3S-FCS-CS				% Cost deviation
S_1	h_v	n_1	n_2	T	TC	$TC+MC$	n	T	TC	$TC+MC$	
10	2	4	1	0.600	2,856	18,856	1	0.162	2,592	22,592	20%
10	2.5	4	1	0.588	2,908	18,908	1	0.160	2,632	22,632	20%
10	3	4	1	0.578	2,960	18,960	1	0.157	2,672	22,672	20%
10	3.5	4	1	0.568	3,010	19,010	1	0.155	2,711	22,711	19%
10	4	4	1	0.558	3,060	19,060	1	0.153	2,750	22,750	19%
20	2	4	1	0.603	2,867	18,867	1	0.166	2,653	22,653	20%
20	2.5	4	1	0.591	2,919	18,919	1	0.163	2,694	22,694	20%
20	3	4	1	0.581	2,971	18,971	1	0.161	2,735	22,735	20%
20	3.5	4	1	0.571	3,021	19,021	1	0.159	2,775	22,775	20%
20	4	4	1	0.561	3,071	19,071	1	0.156	2,814	22,814	20%
200	2	3	1	0.631	3,311	19,311	2	0.365	3,286	23,286	21%
200	2.5	3	3	0.751	3,369	19,369	2	0.360	3,332	23,332	20%
200	3	4	3	0.768	3,409	19,409	2	0.355	3,376	23,376	20%
200	3.5	4	3	0.760	3,448	19,448	2	0.351	3,421	23,421	20%
200	4	4	3	0.752	3,487	19,487	2	0.346	3,464	23,464	20%
1000	2	5	5	1.267	4,149	20,149	5	0.967	5,997	25,997	29%
1000	2.5	5	5	1.254	4,193	20,193	6	0.993	6,042	26,042	29%
1000	3	5	5	1.242	4,237	20,237	6	0.986	6,083	26,083	29%
1000	3.5	5	5	1.229	4,281	20,281	6	0.980	6,124	26,124	29%
1000	4	5	5	1.218	4,324	20,324	6	0.973	6,164	26,164	29%
2000	2	6	6	1.571	4,861	20,861	6	1.143	6,476	26,476	27%
2000	2.5	6	6	1.557	4,907	20,907	7	1.165	6,521	26,521	27%
2000	3	6	6	1.543	4,952	20,952	7	1.158	6,563	26,563	27%
2000	3.5	6	6	1.530	4,997	20,997	7	1.151	6,604	26,604	27%
2000	4	6	6	1.517	5,041	21,041	7	1.144	6,645	26,645	27%

5.1.4. **Vendor's holding cost of returned products and remanufacturing setup cost impact.** For the vendor's holding cost of returned products and the remanufacturing setup cost, it is clearly shown in Table 28 that increasing the retuned products holding cost and the remanufacturing setup cost results in decreasing the savings obtained from the CLSC system, which is anticipated. This reduction in the savings is due to the increased TC and MC for the CLSC compared to the fixed TC and MC of the FCS.

Table 28: Percentage cost deviation for different vendor's holding cost of returned products and remanufacturing setup cost values

		3S-CLSC-CS					% Cost deviation
S_2	h_r	n_1	n_2	T	TC	$TC + MC$	
12.5	0.25	3	1	0.551	2,921	18,921	24%
12.5	0.5	3	1	0.545	2,954	18,954	23%
12.5	1	3	1	0.533	3,019	19,019	23%
12.5	1.5	3	1	0.522	3,083	19,083	22%
12.5	2	3	1	0.512	3,146	19,146	22%
25	0.25	3	1	0.555	2,943	18,943	23%
25	0.5	3	1	0.549	2,977	18,977	23%
25	1	3	1	0.537	3,042	19,042	23%
25	1.5	3	1	0.526	3,107	19,107	22%
25	2	3	1	0.516	3,170	19,170	22%
250	0.25	4	3	0.800	3,263	19,263	21%
250	0.5	4	3	0.789	3,312	19,312	21%
250	1	4	3	0.768	3,409	19,409	20%
250	1.5	3	3	0.723	3,501	19,501	20%
250	2	3	1	0.582	3,576	19,576	19%
1250	0.25	5	5	1.244	4,233	20,233	16%
1250	0.5	5	5	1.223	4,307	20,307	15%
1250	1	5	5	1.183	4,452	20,452	14%
1250	1.5	4	4	1.066	4,592	20,592	14%
1250	2	4	1	0.831	4,665	20,665	13%
2500	0.25	6	6	1.593	5,105	21,105	11%
2500	0.5	6	6	1.564	5,200	21,200	10%
2500	1	6	6	1.511	5,386	21,386	9%
2500	1.5	6	6	1.463	5,566	21,566	8%
2500	2	5	5	1.353	5,737	21,737	8%

In addition, the high values of S_2 result in a bigger reduction in the savings compared to the other tested values. However, operating under CLSC is still more economically advantageous and it achieves an average saving of 18% from all reported results. In addition to that, it is worthy to mention that for all problem instances the CLSC system is generating savings and turns out to be a more economical option than the FSC system that has only one solution for all different values of S_2 and h_r (i.e. $n = 2, T = 0.355, TC = 3{,}376 \; and \; TC + MC = \$23{,}376$).

5.1.5. **Buyer's ordering and holding cost impact.** With regard to the buyer's ordering and holding cost, the results reported in Table 29 show that the holding cost of the buyer has a negligible impact on the savings for the low values of O_b (< 150) and this impact disappears when O_b gets higher.

Table 29: Percentage cost deviation for different buyer's ordering and holding cost values

		3S-CLSC-CS					3S-FCS-CS				% Cost deviation
O_b	h_b	n_1	n_2	T	TC	$TC + MC$	n	T	TC	$TC + MC$	
5	3	11	13	0.762	2,018	18,018	10	0.401	2,245	22,245	23%
5	3.5	11	13	0.734	2,097	18,097	10	0.383	2,353	22,353	24%
5	4	11	13	0.708	2,174	18,174	10	0.370	2,456	22,456	24%
5	4.5	11	13	0.681	2,251	18,251	10	0.350	2,554	22,554	24%
5	5	11	13	0.663	2,319	18,319	10	0.340	2,650	22,650	24%
10	3	8	9	0.764	2,144	18,144	7	0.401	2,346	22,346	23%
10	3.5	8	9	0.734	2,229	18,229	7	0.382	2,458	22,458	23%
10	4	8	9	0.708	2,311	18,311	7	0.366	2,565	22,565	23%
10	4.5	8	9	0.685	2,390	18,390	7	0.352	2,668	22,668	23%
10	5	8	9	0.663	2,466	18,466	7	0.340	2,767	22,767	23%
100	3	4	3	0.834	3,151	19,151	2	0.387	3,098	23,098	21%
100	3.5	4	3	0.799	3,283	19,283	2	0.370	3,240	23,240	21%
100	4	4	3	0.768	3,409	19,409	2	0.355	3,376	23,376	20%
100	4.5	4	3	0.740	3,531	19,531	2	0.342	3,507	23,507	20%
100	5	4	3	0.716	3,649	19,649	2	0.330	3,633	23,633	20%
150	3	5	1	0.794	3,429	19,429	2	0.418	3,347	23,347	20%
150	3.5	5	1	0.762	3,578	19,578	2	0.400	3,500	23,500	20%
150	4	5	1	0.733	3,721	19,721	2	0.384	3,647	23,647	20%
150	4.5	5	1	0.708	3,858	19,858	2	0.370	3,788	23,788	20%
150	5	5	1	0.685	3,991	19,991	2	0.357	3,924	23,924	20%
200	3	4	1	0.814	3,598	19,598	2	0.447	3,578	23,578	20%
200	3.5	4	1	0.786	3,720	19,720	2	0.428	3,742	23,742	20%
200	4	4	1	0.758	3,864	19,864	2	0.410	3,899	23,899	20%
200	4.5	4	1	0.732	4,003	20,003	2	0.395	4,050	24,050	20%
200	5	4	1	0.708	4,137	20,137	2	0.381	4,195	24,195	20%

However, increasing the ordering cost value has a negative impact on the savings obtained, as this results in a slight decrease in the generated savings. On average, we are 21% lower in cost when operating under manufacturing-remanufacturing policy as compared to manufacturing-only policy.

5.1.6. **Raw material ordering and holding costs impact.** The results of the analysis pertaining to the raw material related cost are shown in Table 30 where it is evident that the manufacturing-remanufacturing policy always outperforms the manufacturing-only policy, with savings averaging around 21%.

Table 30: Percentage cost deviation for different raw material ordering and holding cost values

		3S-CLSC-CS					3S-FCS-CS				% Cost deviation
O_v	h_{rm}	n_1	n_2	T	TC	$TC+MC$	n	T	TC	$TC+MC$	
100	1	5	1	0.652	3,116	19,116	2	0.354	2,828	22,828	19%
100	1.5	5	1	0.642	3,160	19,160	2	0.343	2,915	22,915	20%
100	2	3	1	0.576	3,265	19,265	2	0.324	3,082	23,082	20%
100	2.5	4	3	0.792	3,311	19,311	2	0.316	3,162	23,162	20%
100	3	4	3	0.784	3,349	19,349	2	0.309	3,240	23,240	20%
200	1	4	3	0.800	3,263	19,263	2	0.333	3,000	23,000	19%
200	1.5	4	3	0.784	3,337	19,337	2	0.376	3,194	23,194	20%
200	2	4	3	0.768	3,409	19,409	2	0.355	3,376	23,376	20%
200	2.5	3	3	0.727	3,479	19,479	2	0.346	3,464	23,464	20%
200	3	3	1	0.588	3,541	19,541	2	0.338	3,550	23,550	21%
400	1	4	4	0.943	3,496	19,496	2	0.327	3,266	23,266	19%
400	1.5	4	3	0.840	3,576	19,576	2	0.338	3,550	23,550	20%
400	2	4	3	0.823	3,653	19,653	3	0.465	3,873	23,873	21%
400	2.5	4	3	0.807	3,728	19,728	3	0.451	3,987	23,987	22%
400	3	4	3	0.792	3,803	19,803	2	0.390	4,099	24,099	22%
800	1	4	4	1.051	3,896	19,896	2	0.330	3,633	23,633	19%
800	1.5	4	4	1.027	3,990	19,990	2	0.327	4,082	24,082	20%
800	2	4	4	1.004	4,081	20,081	2	0.344	4,648	24,648	23%
800	2.5	4	4	0.982	4,171	20,171	3	0.543	4,792	24,792	23%
800	3	4	4	0.962	4,258	20,258	3	0.528	4,926	24,926	23%
2000	1	4	4	0.957	4,697	20,697	2	0.352	5,109	25,109	21%
2000	1.5	6	6	1.462	4,842	20,842	2	0.377	5,657	25,657	23%
2000	2	6	6	1.425	5,082	21,082	3	0.493	6,083	26,083	24%
2000	2.5	6	6	1.391	5,195	21,195	3	0.466	6,442	26,442	25%
2000	3	6	6	1.359	5,296	21,296	5	0.802	6,735	26,735	26%

It is also noted that increasing the holding cost of raw materials results in slight increase in the percentage of savings obtained. The ordering cost also follows the same trend and slightly increases the savings when it gets higher, keeping all other problem parameters unchanged.

5.2. Effect of Incorporating the Raw Material Stage (3rd Stage)

This section illustrates the impact of incorporating the raw material replenishment decisions as an integral part of the classical two-stage supply chain system, where now procurement, production, and dispatching decisions are simultaneously optimized.

In essence, the comparison is made between the 3S-CLSC-CS model, developed throughout this work, and the 2S-CLSC-CS model developed by Hariga et al. [24]. For the sake of making a fair comparison, the same settings for the problem parameters used in the sensitivity analysis section of Chapter 4 are used here for both models, and the optimal solution for each instance is reported including the optimal production sequence. At the end, the effect of accounting for the raw material stage on the production sequence, lot sizes, cycle time length, production frequencies, and dispatching policy is analyzed.

5.2.1. **Return rate impact.** It is clearly seen in Table 31 that the two-stage and three-stage supply chain systems gave exactly the same optimal production sequence for all given values of the return rate (r). Both models start the production by the remanufacturing process first followed by the manufacturing process (R, M) for all problem instances. This indicates that the production sequence is not being affected by the value of the return rate when comparing the two-stage to the three-stage closed-loop supply chain systems.

In addition, it is evident that optimizing raw material ordering policy, along with other decisions across the chain, does result in either equal or higher number of batches compared to the two-stage system. In the case when the total number of production batches is the same for both, production frequencies turned out to be different between the two systems in the sense that the optimal number of newly manufactured (n_1) and remanufactured (n_2) batches are not the same when comparing the two models. Only one problem instance resulted in the same production frequencies ($r = 500$).

Furthermore, we can see that when total number of production batches is the same for both the two and three sage systems, the 3S-CLSC-CS model results in lower cycle time than the 2S-CLSC-CS.

Table 31: Return rate impact on the operational policy

	3S-CLSC-CS					2S-CLSC-CS				
r	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
400	4	2	0.671	(R,M)	3,809	2	1	0.514	(R,M)	2,919
500	3	2	0.647	(R,M)	3,701	3	2	0.659	(R,M)	2,883
600	4	3	0.566	(R,M)	3,605	2	2	0.592	(R,M)	2,870
700	4	3	0.584	(R,M)	3,490	2	2	0.607	(R,M)	2,800
800	4	3	0.768	(R,M)	3,409	3	4	0.832	(R,M)	2,763
900	4	4	0.859	(R,M)	3,303	2	3	0.705	(R,M)	2,694
1000	3	3	0.752	(R,M)	3,192	2	4	0.804	(R,M)	2,612
1100	3	4	0.855	(R,M)	3,068	2	5	0.893	(R,M)	2,575

5.2.2. **Manufacturing and remanufacturing setup costs impact.** The results in Tables 34 and 35 clearly show that the two models respond differently towards the optimal production sequence for some of the tested instances. Based on the optimal solutions obtained for of the 3S-CLSC-CS model under different values of S_1 and S_2, sequences (M, R) and (R, M) turned out to be the optimal sequence in 28% and 60%, respectively, of all problem instances. However, under 2S-CLSC-CS model, (M, R) sequence did not appear in any of the instances while (R, M) sequence appeared in 84% of the problem instances. Overall, 60% of all problem instances gave exactly the same production sequence for both models and the remaining 40% were different between the 3S-CLSC-CS and 2S-CLSC-CS models under the same settings of the setup cost parameters, where the latter ones are shown in boldface in Table 32.

Moreover, one can see that the total number of production batches is different between the two systems in 56% (14 out of 25) of all the reported iterations. The 3S-CLSC-CS resulted in higher total number of production batches in 10 instances out of the total 25 problem instances. In contrast, the 2S-CLSC-CS resulted in higher total number of production batches in 4 solutions out of the total 25 problem instances. Furthermore, we can see that, for most problem instances, there is a slight difference in the cycle time when incorporating the raw material stage compared to that when raw material costs are not accounted for in the centralized decision making process.

Table 32: Impact of manufacturing and remanufacturing setup costs on the operational policy

		3S-CLSC-CS					2S-CLSC-CS				
S_1	S_2	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
10	12.5	1	1	0.245	(R,M)	2,634	1	1	0.241	(R,M)	1,844
10	**25**	**3**	**1**	**0.471**	**(M,R)**	**2,669**	**1**	**1**	**0.248**	**(R,M)**	**1,895**
10	**250**	**4**	**1**	**0.578**	**(M,R)**	**2,960**	**3**	**2**	**0.630**	**(12211)**	**2,444**
10	**1250**	**9**	**3**	**1.310**	**(111222 111111)**	**4,238**	**3**	**4**	**1.087**	**(R,M)**	**3,608**
10	**2500**	**6**	**3**	**1.458**	**(112221 111)**	**5,093**	**6**	**4**	**1.558**	**(122221 1111)**	**4,518**
20	**12.5**	**3**	**1**	**0.470**	**(M,R)**	**2,664**	**1**	**1**	**0.247**	**(R,M)**	**1,885**
20	**25**	**3**	**1**	**0.475**	**(M,R)**	**2,690**	**1**	**1**	**0.253**	**(R,M)**	**1,935**
20	**250**	**4**	**1**	**0.581**	**(M,R)**	**2,971**	**3**	**2**	**0.638**	**(12211)**	**2,476**
20	1250	4	4	1.056	(R,M)	4,295	3	4	1.089	(R,M)	3,617
20	**2500**	**6**	**3**	**1.462**	**(112221 111)**	**5,107**	**6**	**4**	**1.562**	**(122221 1111)**	**4,531**
200	**12.5**	**3**	**1**	**0.533**	**(M,R)**	**3,019**	**2**	**3**	**0.594**	**(R,M)**	**2,397**
200	**25**	**3**	**1**	**0.537**	**(M,R)**	**3,042**	**2**	**3**	**0.600**	**(R,M)**	**2,418**
200	250	4	3	0.768	(R,M)	3,409	3	4	0.832	(R,M)	2,763
200	1250	5	5	1.288	(R,M)	4,394	3	4	1.138	(R,M)	3,778
200	2500	7	7	1.703	(R,M)	5,232	6	8	1.769	(R,M)	4,635
1000	12.5	4	4	0.994	(R,M)	4,044	3	4	1.016	(R,M)	3,372
1000	25	4	4	0.998	(R,M)	4,056	3	4	1.019	(R,M)	3,384
1000	250	5	5	1.242	(R,M)	4,237	3	4	1.084	(R,M)	3,598
1000	1250	6	6	1.543	(R,M)	4,952	5	7	1.580	(R,M)	4,366
1000	2500	7	7	1.919	(R,M)	5,670	6	8	1.934	(R,M)	5,067
2000	12.5	6	6	1.495	(R,M)	4,797	4	5	1.383	(R,M)	4,212
2000	25	6	6	1.497	(R,M)	4,805	5	7	1.528	(R,M)	4,221
2000	250	6	6	1.543	(R,M)	4,952	5	7	1.580	(R,M)	4,366
2000	1250	7	7	1.804	(R,M)	5,543	6	8	1.884	(R,M)	4,936
2000	2500	7	7	2.086	(R,M)	6,164	6	8	2.122	(R,M)	5,560

5.2.3. **Vendor's holding cost of returned products and remanufacturing setup cost impact.** The results in Table 33 show that the optimal production sequence is not the same for many problem instances under the two models. Based on the optimal solutions obtained from the 3S-CLSC-CS and 2S-CLSC-CS models, (M, R) sequence was the optimal solution in 48% and 8% of the obtained optimal sequence of all problem instances, respectively. On the other hand, (R, M) sequence turned out to be optimal in 52% and 92% of all problem instances when operating under 3S-CLSC-CS and 2S-CLSC-CS, respectively. It is worthy to mention that 60% of all problem instances gave exactly the same production sequence for both models and the other 40% were different between the 3S-CLSC-CS and 2S-CLSC-CS models for the same cost parameters values, where the latter are shown in boldface in Table 33 below.

Table 33: Impact of vendor's holding cost of returned products and remanufacturing setup cost on the operational policy

		3S-CLSC-CS					2S-CLSC-CS				
S_2	h_r	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
12.5	**0.25**	**3**	**1**	**0.551**	**(M,R)**	**2,921**	**2**	**3**	**0.623**	**(R,M)**	**2,288**
12.5	**0.5**	**3**	**1**	**0.545**	**(M,R)**	**2,954**	**2**	**3**	**0.613**	**(R,M)**	**2,325**
12.5	**1**	**3**	**1**	**0.533**	**(M,R)**	**3,019**	**2**	**3**	**0.594**	**(R,M)**	**2,397**
12.5	**1.5**	**3**	**1**	**0.522**	**(M,R)**	**3,083**	**2**	**3**	**0.577**	**(R,M)**	**2,468**
12.5	2	3	1	0.512	(M,R)	3,146	2	1	0.406	(M,R)	2,525
25	**0.25**	**3**	**1**	**0.555**	**(M,R)**	**2,943**	**2**	**3**	**0.628**	**(R,M)**	**2,308**
25	**0.5**	**3**	**1**	**0.549**	**(M,R)**	**2,977**	**2**	**3**	**0.618**	**(R,M)**	**2,345**
25	**1**	**3**	**1**	**0.537**	**(M,R)**	**3,042**	**2**	**3**	**0.600**	**(R,M)**	**2,418**
25	**1.5**	**3**	**1**	**0.526**	**(M,R)**	**3,107**	**2**	**3**	**0.583**	**(R,M)**	**2,489**
25	2	3	1	0.516	(M,R)	3,170	2	1	0.411	(M,R)	2,556
250	0.25	4	3	0.800	(R,M)	3,263	3	4	0.881	(R,M)	2,609
250	0.5	4	3	0.789	(R,M)	3,312	3	4	0.864	(R,M)	2,662
250	1	4	3	0.768	(R,M)	3,409	3	4	0.832	(R,M)	2,763
250	1.5	3	3	0.723	(R,M)	3,501	2	3	0.667	(R,M)	2,849
250	**2**	**3**	**1**	**0.582**	**(M,R)**	**3,576**	**2**	**3**	**0.649**	**(R,M)**	**2,928**
1250	0.25	5	5	1.361	(R,M)	4,154	4	5	1.323	(R,M)	3,553
1250	0.5	5	5	1.335	(R,M)	4,236	4	5	1.294	(R,M)	3,632
1250	1	5	5	1.288	(R,M)	4,394	3	4	1.138	(R,M)	3,778
1250	1.5	5	5	1.245	(R,M)	4,547	3	4	1.099	(R,M)	3,913
1250	**2**	**4**	**1**	**0.831**	**(M,R)**	**4,665**	**3**	**4**	**1.064**	**(R,M)**	**4,042**
2500	0.25	6	7	1.796	(R,M)	4,924	6	8	1.905	(R,M)	4,305
2500	0.5	7	7	1.773	(R,M)	5,022	6	8	1.856	(R,M)	4,418
2500	1	7	7	1.703	(R,M)	5,232	6	8	1.769	(R,M)	4,635
2500	1.5	6	6	1.574	(R,M)	5,427	5	7	1.612	(R,M)	4,839
2500	2	6	6	1.522	(R,M)	5,614	5	7	1.551	(R,M)	5,029

Additionally, we can clearly see that 18 problem instances out of all 25 reported instances (72%) resulted in a different total number of production batches (n) between the two and three stage systems. The 3S-CLSC-CS resulted in higher total number of production batches in 28% of the total 25 problem instances while the 2S-CLSC-CS resulted in higher total number of production batches in 44% of the total 25 problem instances. Moreover, it is clearly noted in the below table that the two models are resulting in different cycle times for all tested values of S_2 and h_r.

5.2.4. **Vendor's holding cost of finished products and manufacturing setup cost impact.** The results in Tables 34 and 35 clearly show that under the same values of the h_v and S_1 cost parameters, the two models respond differently with respect to the optimal production sequence. Under 3S-CLSC-CS model, (M, R) and (R, M) sequences were the optimal production sequences in 44% and 56% of all problem instances, respectively. On the other hand, when operating under 2S-CLSC-CS model, the optimal sequence was to start the production by the remanufacturing process first followed by the manufacturing process (R, M) in 60% of all problem instances. In the other 40% of the instances, the optimal sequence was to start the production by one newly manufactured batch, followed by two consecutive remanufactured batches and again followed by two consecutive newly manufactured batches. Over all problem instances, 54% of them gave exactly the same production sequence for both models and the other 46% were different between the 3S-CLSC-CS and 2S-CLSC-CS models for the same values of the cost parameters.

It is worth mentioning that the three-stage model resulted in higher total number of production batches (n) in 7 instances out of the total 25 reported results. Additionally, 64% of all reported results gave the same total number of production batches. However, the production frequencies (n_1, n_2) for all of them are different Furthermore, is has been noticed that the 2S-CLSC-CS model is always resulting in larger cycle time in all the tested instances except for those when $S_1 = 1000$ where the 3S-CLSC-CS resulted in a larger cycle time.

Table 34: Impact of vendor's holding cost and manufacturing setup cost on the operational policy

		3S-CLSC-CS					2S-CLSC-CS				
S_1	h_v	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
10	**2**	**4**	**1**	**0.600**	**(M,R)**	**2,856**	**3**	**2**	**0.654**	**(12211)**	**2,355**
10	**2.5**	**4**	**1**	**0.588**	**(M,R)**	**2,908**	**3**	**2**	**0.642**	**(12211)**	**2,400**
10	**3**	**4**	**1**	**0.578**	**(M,R)**	**2,960**	**3**	**2**	**0.630**	**(12211)**	**2,444**
10	**3.5**	**4**	**1**	**0.568**	**(M,R)**	**3,010**	**3**	**2**	**0.619**	**(12211)**	**2,488**
10	**4**	**4**	**1**	**0.558**	**(M,R)**	**3,060**	**3**	**2**	**0.608**	**(12211)**	**2,531**
20	**2**	**4**	**1**	**0.603**	**(M,R)**	**2,867**	**3**	**2**	**0.662**	**(12211)**	**2,385**
20	**2.5**	**4**	**1**	**0.591**	**(M,R)**	**2,919**	**3**	**2**	**0.650**	**(12211)**	**2,431**
20	**3**	**4**	**1**	**0.581**	**(M,R)**	**2,971**	**3**	**2**	**0.638**	**(12211)**	**2,476**
20	**3.5**	**4**	**1**	**0.571**	**(M,R)**	**3,021**	**3**	**2**	**0.627**	**(12211)**	**2,520**
20	**4**	**4**	**1**	**0.561**	**(M,R)**	**3,071**	**3**	**2**	**0.616**	**(12211)**	**2,564**

Table 35: Impact of vendor's holding cost and manufacturing setup cost on the operational policy (continued)

		3S-CLSC-CS					2S-CLSC-CS				
S_1	h_v	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
200	**2**	**3**	**1**	**0.631**	**(M,R)**	**3,311**	**2**	**3**	**0.712**	**(R,M)**	**2,668**
200	2.5	3	3	0.751	(R,M)	3,369	2	3	0.699	(R,M)	2,719
200	3	4	3	0.768	(R,M)	3,409	3	4	0.832	(R,M)	2,763
200	3.5	4	3	0.760	(R,M)	3,448	3	4	0.820	(R,M)	2,805
200	4	4	3	0.752	(R,M)	3,487	3	4	0.808	(R,M)	2,845
1000	2	5	5	1.267	(R,M)	4,149	3	4	1.118	(R,M)	3,488
1000	2.5	5	5	1.254	(R,M)	4,193	3	4	1.101	(R,M)	3,544
1000	3	5	5	1.242	(R,M)	4,237	3	4	1.084	(R,M)	3,598
1000	3.5	5	5	1.229	(R,M)	4,281	3	4	1.068	(R,M)	3,652
1000	4	5	5	1.218	(R,M)	4,324	3	4	1.053	(R,M)	3,705
2000	2	6	6	1.571	(R,M)	4,861	4	5	1.476	(R,M)	4,268
2000	2.5	6	6	1.557	(R,M)	4,907	5	7	1.598	(R,M)	4,319
2000	3	6	6	1.543	(R,M)	4,952	5	7	1.580	(R,M)	4,366
2000	3.5	6	6	1.530	(R,M)	4,997	5	7	1.564	(R,M)	4,412
2000	4	6	6	1.517	(R,M)	5,041	6	8	1.638	(R,M)	4,456

5.2.5. **Buyer's Ordering and holding cost impact.** Similar to all previous results, it is clearly seen in Table 36 that the optimal production sequence is not the same for many problem instances of the two models. Based on the optimal solutions obtained from the 3S-CLSC-CS model, (M, R) and (R, M) sequences were the optimal solution of 40% and 60% of the obtained optimal sequences of all problem instances, respectively. In contrast, when operating under 2S-CLSC-CS model, the optimal sequence for all problem instances was to start the production by the remanufacturing process followed by the manufacturing process (R, M). The difference in the sequence under both models appears for values of the ordering cost that are greater than or equal to 150.

Furthermore, it is clearly seen that adding the raw material stage to the integrated supply chain results in either equal or higher number of batches compared to the two-stage system. In the case when the total number of production batches is the same for both, production frequencies turned out to be different between the two systems. In addition to that, we can see that when $O_b < 100$, the 3S-CLSC-CS resulted in slightly larger cycle time than the 2S-CLSC-CS model. However, when $O_b \geq 100$, the 2S-CLSC-CS resulted in slightly larger cycle time than the 3S-CLSC-CS model.

Table 36: Impact of buyer's ordering and holding cost on the operational policy

		3S-CLSC-CS					2S-CLSC-CS				
O_b	h_b	n_1	n_2	T	Sequence	TC	n_1	n_2	T	Sequence	TC
5	3	11	13	0.762	(R,M)	2,018.1	9	12	0.753	(R,M)	1,475.0
5	3.5	11	13	0.734	(R,M)	2,097.3	9	12	0.715	(R,M)	1,553.2
5	4	11	13	0.708	(R,M)	2,173.7	9	12	0.682	(R,M)	1,627.6
5	4.5	11	13	0.681	(R,M)	2,250.9	9	12	0.653	(R,M)	1,698.8
5	5	11	13	0.663	(R,M)	2,318.8	9	12	0.628	(R,M)	1,767.1
10	3	8	9	0.764	(R,M)	2,144.3	6	8	0.739	(R,M)	1,596.5
10	3.5	8	9	0.734	(R,M)	2,229.2	6	8	0.703	(R,M)	1,679.3
10	4	8	9	0.708	(R,M)	2,311.0	6	8	0.671	(R,M)	1,758.3
10	4.5	8	9	0.685	(R,M)	2,389.9	6	8	0.643	(R,M)	1,833.8
10	5	8	9	0.663	(R,M)	2,466.3	6	8	0.619	(R,M)	1,906.4
100	3	4	3	0.834	(R,M)	3,151.0	3	4	0.913	(R,M)	2,519.5
100	3.5	4	3	0.799	(R,M)	3,282.5	3	4	0.870	(R,M)	2,644.2
100	4	4	3	0.768	(R,M)	3,409.0	3	4	0.832	(R,M)	2,763.3
100	4.5	4	3	0.740	(R,M)	3,530.9	3	4	0.799	(R,M)	2,877.5
100	5	4	3	0.716	(R,M)	3,648.8	3	4	0.770	(R,M)	2,987.3
150	**3**	**5**	**1**	**0.794**	**(M,R)**	**3,429.4**	**2**	**3**	**0.845**	**(R,M)**	**2,839.7**
150	**3.5**	**5**	**1**	**0.762**	**(M,R)**	**3,578.1**	**2**	**3**	**0.806**	**(R,M)**	**2,978.6**
150	**4**	**5**	**1**	**0.733**	**(M,R)**	**3,720.9**	**2**	**3**	**0.771**	**(R,M)**	**3,111.3**
150	**4.5**	**5**	**1**	**0.708**	**(M,R)**	**3,858.4**	**2**	**3**	**0.741**	**(R,M)**	**3,238.5**
150	**5**	**5**	**1**	**0.685**	**(M,R)**	**3,991.1**	**2**	**3**	**0.714**	**(R,M)**	**3,361.0**
200	**3**	**4**	**1**	**0.814**	**(M,R)**	**3,597.5**	**2**	**3**	**0.929**	**(R,M)**	**3,121.5**
200	**3.5**	**4**	**1**	**0.786**	**(M,R)**	**3,719.5**	**2**	**3**	**0.886**	**(R,M)**	**3,274.2**
200	**4**	**4**	**1**	**0.758**	**(M,R)**	**3,863.8**	**2**	**3**	**0.848**	**(R,M)**	**3,420.0**
200	**4.5**	**4**	**1**	**0.732**	**(M,R)**	**4,003.0**	**2**	**3**	**0.815**	**(R,M)**	**3,559.9**
200	**5**	**4**	**1**	**0.708**	**(M,R)**	**4,137.4**	**2**	**3**	**0.785**	**(R,M)**	**3,694.5**

5.3. Conclusion

To conclude this chapter, it is important to mention that it is more economically advantageous to have a CLSC than a FSC for a three-stage supply chain system. Results show that increasing the return rate (r), manufacturing setup cost (S_1), raw material ordering cost (O_v) and holding cost of raw material (h_{rm}) results in higher savings and drive the CLSC system to be more economical. In contrast, all other tested problem parameters result in decreasing the savings generated when their values get higher. Furthermore, it is worthy to mention that in all the 113 reported instances, the 3S-CLSC-CS system achieved, on average, 21% of cost reduction when compared to the 3S-FSC-CS system.

Moreover, the results reported in Tables 31 to 36 show that adding the raw material stage does have an impact not only on the optimal production sequence, but also on the cycle time (T) as well as the production frequencies (n_1, n_2). It turned out that (M, R) and (R, M) production sequences were the optimal sequence in 37% and 60% of all 108 problem instances, respectively, when operating under the 3S-CLSC-CS model. In contrast, these percentages changed to 2% and 85%, respectively, when operating under the 2S-CLSC-CS model. Hence, a noticeable change in (M, R) and (R, M) sequences can be found in the optimal solutions when adding/removing the raw material stage. Overall, it is worthy to mention that 62% of all problem instances gave exactly the same production sequence for both models, with the remaining 38% being different between the two models under values for cost parameters. Furthermore, it has been found that the total number of production batches (n) turned out to be different between the 2S-CLSC-CS and 3S-CLSC-CS systems in 56% (60 out of 108) of all reported iterations. In addition, the production frequencies (n_1, n_2) were the same for both models only twice in all of the 108 reported optimal solutions. Moreover, the cycle time found to be different for all problem instances even when production frequencies were the same.

Chapter 6: Conclusion and Future Research Directions

6.1. Conclusion

World class organizations believe that working only within the company's borders, in complete isolation of the suppliers and the customers does not yield the anticipated success or the desired optimality of the operational decisions made. As such, a more emphasis is placed on having overall system efficiency and being globally optimum rather than achieving individual efficiency and local optimality. The objective is to continuously improve their internal processes as well as closely collaborating with supply chain partners towards achieving better service levels at a reduced chain-wide total cost. One form of strategic partnership is consignment stock (CS) in which the stock is kept close to the point of consumption at the buyer's premises, while still owned by the vendor, until it has been sold to the end use at which point in time the vendor gets paid for the sold items. Such form of partnership is a widely accepted practice in the retailing industry, and it has proved to be beneficial for the vendor and buyer alike. From a research perspective, integrated supply chain models operating under CS partnership have received a great attention in an attempt to better synchronize the operational decisions and thus optimize the chain-wide total cost. On another note, ample research works have shown that adopting the environmentally friendly approach concerning the efficient management and retrieval of products might result in cost savings, and also achieve sustainability related objectives.

A thorough examination of the literature reveals that there have been several researchers who addressed the optimization of production and inventory decisions in the context of forward three-stage (3S) supply chain systems. However, the developed models were mostly restrictive when it comes to the adopted replenishment policy of raw material, as prior simplifying assumptions of such policy were typically made. Moreover, it is noted that integrated inventory models under CS partnership have been developed for single and two stage supply chain systems. In addition, there is only one research work that dealt with a three-stage closed-loop supply chain model [41]. However, in this work, no CS agreement was followed between the vendor and buyer, and raw material replenishment policy was restricted to be only one order per cycle and just before the manufacturing process, and that the sequencing of the batches was predetermined. Thus, it is evident that there is a need for a research work that takes into consideration CS partnership in the context of three-stage forward and CLSC systems, where it also allows for the optimization of raw material replenishment decisions with no prior restriction, as well as a generalized model that optimizes the sequencing of newly manufactured and remanufactured batches in the case of CLSS system.

The key contribution of this work is the development of mixed integer non-linear programming (MINLP) models that jointly seek to optimize the procurement decisions of raw material, the length of the production cycle, the sequence to follow in the production of newly and remanufactured batches, number of newly and remanufactured batches produced within one production cycle, as well as the initial inventory levels of recovered and finished products at the vendor's and buyer's premises, respectively, in order to minimize the chain-wide total cost.

In this thesis, two separate solution procedures were developed to determine the most economic raw material ordering policy as well as the optimum number of manufacturing and remanufacturing batches for the forward and closed loop supply chain optimization problems under consignment stock partnership. Extensive sensitivity analysis was carried out for different problem instances under different settings of the problem parameters to study the impact of changing the cost parameters (holding, setup and ordering costs) on the behavior of the model. Towards validating the importance of the work presented in this thesis, the following comparisons were conducted under different settings of the problem parameters:

1) The forward three stage model was compared to the two-stage model of Braglia and Zavanella [34], in order to analyze the impact of adding raw material replenishment decisions on the adopted lot sizing and dispatching policy.
2) The closed loop three stage CLSC model was compared to the two stage CLSC model presented by Hariga et al. [24] where special attention is paid to the behavior of the resulting production sequence under both models, along with other decisions variables.
3) The three stage CLSC model is compared to the three-stage forward supply chain model to assess the economic gains of the returned products and the remanufacturing process, which is reported as a percentage cost savings.

It is important to point out that the models developed by Braglia and Zavanella [34] and Hariga et al. [24] are special cases of the three stage forward and closed loop models presented in this thesis, respectively. Moreover, the obtained results validated that the closed-loop supply chain turned out to be more economically advantageous than the forward supply chain for a three-stage system. It has been found that, on average, the 3S-CLSC-CS system achieved 21% of cost reduction when compared to the 3S-FSC-CS system. Additionally, results showed that raw material cost components do have a major impact on the related decision variables and hence on the long-run system wide cost for both the forward and closed-loop supply chain systems. Adding the third stage impacts the optimal production sequence as well as the other decision variable (i.e. number of batches, lot sizing and cycle time). It turned out that (M, R) and (R, M) production sequences were the optimal sequence in 37% and 60% of all reported problem instances, respectively, for the three-stage closed loop supply chain system. However, for the two stage CLSC system, these percentages change to 2% and 85% for the (M, R) and (R, M) sequences, respectively. For the three-stage model, the remaining 3% optimal solutions found to be for the intermittent sequences in case of high setup costs for the remanufactured batches and low setup costs for the newly manufactured ones.

6.2. Future Research Directions

The work presented in this thesis stands out as an important contribution to the multi stage supply chain systems which can be further extended in several directions through incorporating additional realistic factors. For instance, extending the present work to deal with stochastic, rather than deterministic, return rate and demand would be a valuable addition in which uncertainties will be taken into account while solving the models. In addition, relaxing the assumption that disposal rate is zero would be another direction of extension to this work as some of the returned items might not be reusable. Incorporating the environmental concerns, such as CO_2 emissions generated from the collection of returned products, the remanufacturing and manufacturing processes into the developed models is a rich topic to further enhance the present work.

Furthermore, an interesting future research avenue is the consideration of a supply chain network involving multiple buyers. A research direction that adds several complexities to the problem however. This includes adding the sequencing of shipments to the buyers along with the optimization of production sequence. Another challenging, yet interesting, research direction that requires further investigation is to optimize the use of the idle time so that it is scheduled across the cycle time rather than postponing it to the end of the cycle. This might have economic advantages depending on the trade-off between the reduction of the buyer's inventory holding cost and the increase in the other inventory holding costs (i.e. raw material and returned products) as well as the manufacturing and remanufacturing setup costs.

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Appendix A- Equations' derivation

- Detailed derivation of equation (22):

$$K(n) = \frac{O_v+nO_b+s}{\sqrt{\frac{O_v+nO_b+s}{(RH_{rm}+RH_v+RH_b)}}} + \left(\sqrt{\frac{O_v+nO_b+s}{(RH_{rm}+RH_v+RH_b)}}\right)(RH_{rm} + RH_v + RH_b)$$

$$= \sqrt{(RH_{rm} + RH_v + RH_b)(O_v + nO_b + \mathrm{s})} + \sqrt{(RH_{rm} + RH_v + RH_b)(O_v + nO_b + \mathrm{s})}$$

$$= 2\sqrt{(RH_{rm} + RH_v + RH_b)(O_v + nO_b + \mathrm{s})}$$

- Detailed derivation of equation (23)

$$2\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + n^*O_b + \mathrm{s})} -$$
$$2\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + (n^*-1)O_b + \mathrm{s})} \leq 0$$

Dividing by 2 and moving the negative part to the other side result in

$$\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + n^*O_b + \mathrm{s})}$$
$$\leq \sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + (n^*-1)O_b + \mathrm{s})}$$

Squaring both sides results in

$$\left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + n^*O_b + \mathrm{s})$$
$$\leq \left(h_{rm}\left(\frac{D^2}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v + (n^*-1)O_b + \mathrm{s})$$

Expanding the brackets results in

$$h_{rm}\left(\frac{D^2}{2P}\right)O_v+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)O_v+\frac{h_bD^2}{2\,n^*\,P}O_v+\;h_bO_v\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)n^*O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{h_bD^2}{2\;P}O_b$$
$$+\;h_bn^*O_b\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{h_bD^2}{2\,n^*\,P}\mathrm{s}+\;h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)$$
$$\leq\;h_{rm}\left(\frac{D^2}{2P}\right)O_v+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)O_v+\frac{h_bD^2}{2\,(n^*-1)\,P}O_v+\;h_bO_v\left(\frac{(P-D)D}{2P}\right)$$
$$+h_{rm}\left(\frac{D^2}{2P}\right)(n^*-1)O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{h_bD^2}{2\;P}O_b+\;h_b(n^*-1)O_b\left(\frac{(P-D)D}{2P}\right)$$
$$+h_{rm}\left(\frac{D^2}{2P}\right)\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)\mathrm{s}+\frac{h_bD^2}{2\,(n^*-1)\,P}\mathrm{s}+\;h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)$$

Removing the repeated terms gives

$$\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)O_v+\frac{h_bD^2}{2\,n^*\,P}O_v+\;h_{rm}\left(\frac{D^2}{2P}\right)n^*O_b+\;h_bn^*O_b\left(\frac{(P-D)D}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{h_bD^2}{2\,n^*\,P}\mathrm{s}$$
$$\leq\;\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)O_v+\frac{h_bD^2}{2\,(n^*-1)\,P}O_v+\;h_{rm}\left(\frac{D^2}{2P}\right)(n^*-1)O_b$$
$$+\;h_b(n^*-1)O_b\left(\frac{(P-D)D}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)\mathrm{s}+\frac{h_bD^2}{2\,(n^*-1)\,P}\mathrm{s}$$

Sorting out the common terms results in

$$\frac{1}{\mathrm{n}^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\;P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\;P}\mathrm{s}\right]+\;\mathrm{n}^*\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\;h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\leq\frac{1}{\mathrm{n}^*-1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\;P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\;P}\mathrm{s}\right]$$
$$+\;(\mathrm{n}^*-1)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\;h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$

Putting all terms on one side gives

$$\frac{1}{\mathrm{n}^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\;P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\;P}\mathrm{s}\right]-\frac{1}{\mathrm{n}^*-1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\;P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\;P}\mathrm{s}\right]$$
$$+\;\mathrm{n}^*\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\;h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]-(\mathrm{n}^*-1)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\;h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\leq\;0$$

Sorting out the common terms results in

$$\left(\frac{1}{\mathrm{n}^*}-\frac{1}{\mathrm{n}^*-1}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\;P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\;P}\mathrm{s}\right]$$
$$+\;\left(\mathrm{n}^*-(\mathrm{n}^*-1)\right)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\;h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]\leq\;0$$

$\mathrm{n}^*-(\mathrm{n}^*-1)=1$, then

$$\left(\frac{1}{n^*}-\frac{1}{n^*-1}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]+\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]\le\ 0$$

Putting the unknowns on one side gives

$$\left(\frac{1}{n^*}-\frac{1}{n^*-1}\right)\ \le\frac{-\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

Doing algebraic manipulations to the left side gives

$$\left(\frac{(n^*-1)}{n^*(n^*-1)}-\frac{n^*}{n^*(n^*-1)}\right)\ \le\frac{-\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

$$\frac{-1}{n^*(n^*-1)}\ \le\frac{-\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

$$\frac{1}{n^*(n^*-1)}\ \ge\frac{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

Taking the reciprocal gives

$$n^*(n^*-1)\ \le\frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}$$

Doing algebraic manipulations to the right side gives

$$n^*(n^*-1)\ \le\frac{\frac{D^2}{2P}\left[h_vO_v+h_bO_v+h_vs+h_bs\right]}{\frac{D^2}{2P}\left[h_{rm}O_b+\ h_bO_b\left(\frac{(P-D)}{D}\right)\right]}$$

$$n^*(n^*-1)\ \le\frac{(h_v+h_b)(O_v+s)}{O_b\left[h_{rm}+\ h_b\left(\frac{(P-D)}{D}\right)\right]}$$

- Detailed derivation of equation (24)

$$2\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\ (n^*+1)}\right)+h_b\left[\frac{D^2}{2\ (n^*+1)\ P}+\ \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+(n^*+1)O_b+s)}-$$

$$2\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\ n^*}\right)+h_b\left[\frac{D^2}{2\ n^*\ P}+\ \left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+n^*O_b+s)}\ \ \ge 0$$

Dividing by 2 and moving the negative part to the other side result in

$$\sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)+h_b\left[\frac{D^2}{2\,(n^*+1)\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+(n^*+1)O_b+\mathrm{s})} \geq \sqrt{\left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)+h_b\left[\frac{D^2}{2\,n^*\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+n^*O_b+\mathrm{s})}$$

Squaring both sides results in

$$\left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)+h_b\left[\frac{D^2}{2\,(n^*+1)\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+(n^*+1)O_b+\mathrm{s}) \geq \left(h_{rm}\left(\frac{D^2}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)+h_b\left[\frac{D^2}{2\,n^*\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)(O_v+n^*O_b+\mathrm{s})$$

Expanding the brackets results in

$$\begin{aligned}
&h_{rm}\left(\frac{D^2}{2P}\right)O_v+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)O_v+\frac{h_bD^2}{2\,(n^*+1)\,P}O_v+h_bO_v\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)(n^*+1)O_b\\
&\quad+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{h_bD^2}{2\;P}O_b+h_b(n^*+1)O_b\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)\mathrm{s}\\
&\quad+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)\mathrm{s}+\frac{h_bD^2}{2\,(n^*+1)\,P}\mathrm{s}+h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)\\
&\quad\geq h_{rm}\left(\frac{D^2}{2P}\right)O_v+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)O_v+\frac{h_bD^2}{2\,n^*\,P}O_v+h_bO_v\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)n^*O_b\\
&\quad+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{h_bD^2}{2\;P}O_b+h_bn^*O_b\left(\frac{(P-D)D}{2P}\right)+h_{rm}\left(\frac{D^2}{2P}\right)\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{h_bD^2}{2\,n^*\,P}\mathrm{s}\\
&\quad+h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)
\end{aligned}$$

Removing the repeated terms gives

$$\begin{aligned}
&\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)O_v+\frac{h_bD^2}{2\,(n^*+1)\,P}O_v+h_{rm}\left(\frac{D^2}{2P}\right)(n^*+1)O_b+h_b(n^*+1)O_b\left(\frac{(P-D)D}{2P}\right)\\
&\quad+\frac{h_v}{2}\left(\frac{D^2}{P\,(n^*+1)}\right)\mathrm{s}+\frac{h_bD^2}{2\,(n^*+1)\,P}\mathrm{s}\\
&\quad\geq\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)O_v+\frac{h_bD^2}{2\,n^*\,P}O_v+h_{rm}\left(\frac{D^2}{2P}\right)n^*O_b+h_bn^*O_b\left(\frac{(P-D)D}{2P}\right)+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}\\
&\quad+\frac{h_bD^2}{2\,n^*\,P}\mathrm{s}
\end{aligned}$$

Sorting out the common terms results in

$$\frac{1}{n^*+1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]+(n^*+1)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\geq\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]$$
$$+\ n^*\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$

Putting all terms on one side gives

$$\frac{1}{n^*+1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]-\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]$$
$$+(n^*+1)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]-\ n^*\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\geq 0$$

Sorting out the common terms results in

$$\left(\frac{1}{n^*+1}-\frac{1}{n^*}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]$$
$$+(n^*+1-\ n^*)\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]\geq 0$$

$n^*+1-n^*=1$, then

$$\left(\frac{1}{n^*+1}-\frac{1}{n^*}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]+\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]\geq 0$$

Putting the unknowns on one side gives

$$\left(\frac{1}{n^*+1}-\frac{1}{n^*}\right)\geq-\frac{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

Doing algebraic manipulations to the left side gives

$$\left(\frac{n^*}{n^*(n^*+1)}-\frac{(n^*+1)}{n^*(n^*+1)}\right)\geq-\frac{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

$$\left(\frac{-1}{n^*(n^*+1)}\right)\geq-\frac{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v+\frac{h_bD^2}{2\ P}O_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)s+\frac{h_bD^2}{2\ P}s\right]}$$

Multiplying both sides by -1 gives

$$\left(\frac{1}{\mathrm{n}^*(\mathrm{n}^*+1)}\right) \leq \frac{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v + \frac{h_b D^2}{2\ P}O_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\ P}\mathrm{s}\right]}$$

Taking the reciprocal gives

$$\mathrm{n}^*(\mathrm{n}^*+1) \geq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_v + \frac{h_b D^2}{2\ P}O_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\ P}\mathrm{s}\right]}{\left[h_{rm}\left(\frac{D^2}{2P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]}$$

Doing algebraic manipulations to the right side gives

$$\mathrm{n}^*(\mathrm{n}^*+1) \geq \frac{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[h_v O_v + h_b O_v + h_v \mathrm{s} + h_b \mathrm{s}\right]}{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[h_{rm}O_b + h_b O_b\left(\frac{(P-D)}{D}\right)\right]}$$

$$\mathrm{n}^*(\mathrm{n}^*+1) \geq \frac{(h_v + h_b)(O_v + \mathrm{s})}{O_b\left[h_{rm} + h_b\left(\frac{(P-D)}{D}\right)\right]}$$

- Detailed derivation of equation (29)

$$K(n,u) = \frac{\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}}{\sqrt{\frac{\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}}{(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)}}T} + \left(\sqrt{\frac{\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}}{(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)}}\right)(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)$$

$$= \sqrt{(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)\left(\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}\right)} + \sqrt{(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)\left(\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}\right)}$$

$$= 2\sqrt{(RH_{rm}^{\mathrm{u}} + RH_v + RH_b)\left(\frac{O_v}{\mathrm{u}} + nO_b + \mathrm{s}\right)}$$

- Detailed derivation of equation (30)

$$2\sqrt{\left(\frac{h_{rm}}{2}\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}} + n^*O_b + \mathrm{s}\right)}$$
$$-2\sqrt{\left(\frac{h_{rm}}{2}\mathrm{D}\left[\left(\frac{D}{P}\right) + (\mathrm{u}-1)\right] + \frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}} + (n^*-1)O_b + \mathrm{s}\right)}$$
$$\leq 0$$

Dividing by 2, moving the negative part to the other side and then squaring both sides result in

$$\left(\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)+h_b\left[\frac{D^2}{2\,n^*\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+n^*O_b+\mathrm{s}\right)$$

$$\leq\left(\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right)\right.$$

$$\left.+h_b\left[\frac{D^2}{2\,(n^*-1)\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+(n^*-1)O_b+\mathrm{s}\right)$$

Expanding the brackets results in

$$\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,n^*\,P}h_b\frac{O_v}{\mathrm{u}}+\left(\frac{(P-D)D}{2P}\right)h_b\frac{O_v}{\mathrm{u}}$$

$$+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]n^*O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{D^2}{2\ P}h_bO_b$$

$$+\left(\frac{(P-D)D}{2P}\right)h_bn^*O_b+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{D^2}{2\,n^*\,P}h_b\mathrm{s}$$

$$+\left(\frac{(P-D)D}{2P}\right)h_b\mathrm{s}$$

$$\leq\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,(n^*-1)\,P}h_b\frac{O_v}{\mathrm{u}}$$

$$+\left(\frac{(P-D)D}{2P}\right)h_b\frac{O_v}{\mathrm{u}}+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right](n^*-1)O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b$$

$$+\frac{D^2}{2\ P}h_bO_b+\left(\frac{(P-D)D}{2P}\right)h_b(n^*-1)O_b+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\mathrm{s}$$

$$+\frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right)\mathrm{s}+\frac{D^2}{2\,(n^*-1)\,P}h_b\mathrm{s}+\left(\frac{(P-D)D}{2P}\right)h_b\mathrm{s}$$

Removing the repeated terms gives

$$\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,n^*\,P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]n^*O_b+\left(\frac{(P-D)D}{2P}\right)h_bn^*O_b$$

$$+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{D^2}{2\,n^*\,P}h_b\mathrm{s}$$

$$\leq\frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,(n^*-1)\,P}h_b\frac{O_v}{\mathrm{u}}$$

$$+\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right](n^*-1)O_b+\left(\frac{(P-D)D}{2P}\right)h_b(n^*-1)O_b$$

$$+\frac{h_v}{2}\left(\frac{D^2}{P(n^*-1)}\right)\mathrm{s}+\frac{D^2}{2\,(n^*-1)\,P}h_b\mathrm{s}$$

Sorting out the common terms results in

$$\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$+n^*\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]$$
$$\le\frac{1}{(n^*-1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$+(n^*-1)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]$$

Putting all terms on one side gives

$$\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$-\frac{1}{(n^*-1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$+n^*\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]$$
$$-(n^*-1)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\le 0$$

Sorting out the common terms results in

$$\left(\frac{1}{n^*}-\frac{1}{(n^*-1)}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$+\left(n^*-(n^*-1)\right)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\le 0$$

$n^*-(n^*-1)=1$, and doing some algebraic manipulations to the left side give

$$\left(\frac{n^*-1}{n^*(n^*-1)}-\frac{n^*}{n^*(n^*-1)}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$+\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\le 0$$

Moving the positive part to the right results in

$$\frac{-1}{n^*(n^*-1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]$$
$$\le-\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]$$

Putting the unknowns on one side gives

$$\frac{-1}{n^*(n^*-1)} \leq -\frac{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\ \left(\frac{(P-D)D}{2P}\right)h_bO_b\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}$$

Multiplying both sides by -1 gives

$$\frac{1}{n^*(n^*-1)} \geq \frac{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\ \left(\frac{(P-D)D}{2P}\right)h_bO_b\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}$$

Taking the reciprocal results in

$$n^*(n^*-1) \leq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\ \left(\frac{(P-D)D}{2P}\right)h_bO_b\right]}$$

Expanding denominator’s brackets results in

$$n^*(n^*-1) \leq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}{\left[\frac{D^2}{2P}\ h_{rm}O_b+\frac{\mathrm{D}}{2}(\mathrm{u}-1)h_{rm}O_b+\ \left(\frac{(P-D)D}{2P}\right)h_bO_b\right]}$$

Doing algebraic manipulations to the right side gives

$$n^*(n^*-1) \leq \frac{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[\frac{O_vh_v}{\mathrm{u}}+h_b\frac{O_v}{\mathrm{u}}+sh_v+h_b\mathrm{s}\right]}{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[\ h_{rm}O_b+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)h_{rm}O_b+\ \left(\frac{(P-D)}{D}\right)h_bO_b\right]}$$

$$n^*(n^*-1) \leq \frac{h_v\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)+h_b\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)}{O_b\left[h_{rm}\left(1+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)\right)+\ \left(\frac{(P-D)}{D}\right)h_b\right]}$$

$$n^*(n^*-1) \leq \frac{\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)(h_v+h_b)}{O_b\left[h_{rm}\left(1+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)\right)+\ \left(\frac{(P-D)}{D}\right)h_b\right]}$$

- Detailed derivation of equation (31)

$$2\sqrt{\left(\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)+h_b\left[\frac{D^2}{2\,(n^*+1)\,P}+\ \left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+(n^*+1)O_b+\mathrm{s}\right)}$$
$$-\,2\sqrt{\left(\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)+h_b\left[\frac{D^2}{2\,n^*\,P}+\ \left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+n^*O_b+\mathrm{s}\right)} \geq 0$$

Dividing by 2, moving the negative part to the other side and then squaring both sides result in

$$\left(\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)+h_b\left[\frac{D^2}{2\,(n^*+1)\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+(n^*+1)O_b+\mathrm{s}\right)$$
$$\geq\left(\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)+h_b\left[\frac{D^2}{2\,n^*\,P}+\left(\frac{(P-D)D}{2P}\right)\right]\right)\left(\frac{O_v}{\mathrm{u}}+n^*O_b+\mathrm{s}\right)$$

Expanding the brackets results in

$$\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,(n^*+1)\,P}h_b\frac{O_v}{\mathrm{u}}+\left(\frac{(P-D)D}{2P}\right)h_b\frac{O_v}{\mathrm{u}}$$
$$+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right](n^*+1)O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{D^2}{2\ P}h_bO_b$$
$$+\left(\frac{(P-D)D}{2P}\right)h_b(n^*+1)O_b+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)\mathrm{s}$$
$$+\frac{D^2}{2\,(n^*+1)\,P}h_b\mathrm{s}+\left(\frac{(P-D)D}{2P}\right)h_b\mathrm{s}$$
$$\geq\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,n^*\,P}h_b\frac{O_v}{\mathrm{u}}$$
$$+\left(\frac{(P-D)D}{2P}\right)h_b\frac{O_v}{\mathrm{u}}+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]n^*O_b+\frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b+\frac{D^2}{2\ P}h_bO_b$$
$$+\left(\frac{(P-D)D}{2P}\right)h_bn^*O_b+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]\mathrm{s}+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{D^2}{2\,n^*\,P}h_b\mathrm{s}$$
$$+\left(\frac{(P-D)D}{2P}\right)h_b\mathrm{s}$$

Removing the repeated terms gives

$$\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,(n^*+1)\,P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right](n^*+1)O_b$$
$$+\left(\frac{(P-D)D}{2P}\right)h_b(n^*+1)O_b+\frac{h_v}{2}\left(\frac{D^2}{P(n^*+1)}\right)\mathrm{s}+\frac{D^2}{2\,(n^*+1)\,P}h_b\mathrm{s}$$
$$\geq\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2\,n^*\,P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]n^*O_b$$
$$+\left(\frac{(P-D)D}{2P}\right)h_bn^*O_b+\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s}+\frac{D^2}{2\,n^*\,P}h_b\mathrm{s}$$

Sorting out the common terms results in

$$
\begin{aligned}
&\frac{1}{(n^*+1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad +(n^*+1)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right] \\
&\qquad \geq \frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad +n^*\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]
\end{aligned}
$$

Putting all terms on one side gives

$$
\begin{aligned}
&\frac{1}{(n^*+1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}\right. \\
&\qquad -\frac{1}{n^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]\left.+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad +(n^*+1)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right] \\
&\qquad -n^*\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\geq 0
\end{aligned}
$$

Sorting out the common terms results in

$$
\begin{aligned}
&\left(\frac{1}{(n^*+1)}-\frac{1}{n^*}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad +\left((n^*+1)-n^*\right)\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\geq 0
\end{aligned}
$$

$(n^*+1)-n^*=1$, and doing some algebraic manipulations to the left side give

$$
\begin{aligned}
&\left(\frac{n^*}{n^*(n^*+1)}-\frac{(n^*+1)}{n^*(n^*+1)}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad +\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]\geq 0
\end{aligned}
$$

Moving the positive part to the right results in

$$
\begin{aligned}
&\frac{-1}{n^*(n^*+1)}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right] \\
&\qquad \geq -\left[\frac{h_{rm}}{2}\,\mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b+\left(\frac{(P-D)D}{2P}\right)h_bO_b\right]
\end{aligned}
$$

Putting the unknowns on one side gives

$$\frac{-1}{n^*(n^*+1)} \geq -\frac{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b + \left(\frac{(P-D)D}{2P}\right)h_b O_b\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}$$

Multiplying both sides by -1 gives

$$\frac{1}{n^*(n^*+1)} \leq \frac{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b + \left(\frac{(P-D)D}{2P}\right)h_b O_b\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}$$

Taking the reciprocal results in

$$n^*(n^*+1) \geq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}{\left[\frac{h_{rm}}{2}\ \mathrm{D}\left[\left(\frac{D}{P}\right)+(\mathrm{u}-1)\right]O_b + \left(\frac{(P-D)D}{2P}\right)h_b O_b\right]}$$

Expanding denominator’s brackets results in

$$n^*(n^*+1) \geq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\frac{O_v}{\mathrm{u}}+\frac{D^2}{2P}h_b\frac{O_v}{\mathrm{u}}+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{D^2}{2P}h_b\mathrm{s}\right]}{\left[\frac{D^2}{2P}\ h_{rm}O_b+\frac{\mathrm{D}}{2}(\mathrm{u}-1)h_{rm}O_b + \left(\frac{(P-D)D}{2P}\right)h_b O_b\right]}$$

Doing algebraic manipulations to the right side gives

$$n^*(n^*+1) \geq \frac{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[\frac{O_v h_v}{\mathrm{u}}+h_b\frac{O_v}{\mathrm{u}}+\mathrm{s}h_v+h_b\mathrm{s}\right]}{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[h_{rm}O_b+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)h_{rm}O_b+\left(\frac{(P-D)}{D}\right)h_b O_b\right]}$$

$$n^*(n^*+1) \geq \frac{h_v\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)+h_b\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)}{O_b\left[h_{rm}\left(1+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)\right)+\left(\frac{(P-D)}{D}\right)h_b\right]}$$

$$n^*(n^*+1) \geq \frac{\left(\frac{O_v}{\mathrm{u}}+\mathrm{s}\right)(h_v+h_b)}{O_b\left[h_{rm}\left(1+\frac{\mathrm{P}}{\mathrm{D}}(\mathrm{u}-1)\right)+\left(\frac{(P-D)}{D}\right)h_b\right]}$$

- Detailed derivation of equation (36)

$$K(n,m) = \frac{\mathrm{m}O_v+nO_b+\mathrm{s}}{\sqrt{\frac{\mathrm{m}O_v+nO_b+\mathrm{s}}{(RH_{rm}^{\mathrm{m}}+RH_v+RH_b)}}T} + \left(\sqrt{\frac{\mathrm{m}O_v+nO_b+\mathrm{s}}{(RH_{rm}^{\mathrm{m}}+RH_v+RH_b)}}\right)(RH_{rm}^{\mathrm{m}}+RH_v+RH_b)$$

$$= \sqrt{(RH_{rm}^{\mathrm{m}} + RH_v + RH_b)(\mathrm{m}O_v + nO_b + \mathrm{s})} + \sqrt{(RH_{rm}^{\mathrm{m}} + RH_v + RH_b)(\mathrm{m}O_v + nO_b + \mathrm{s})}$$

$$= 2\sqrt{(RH_{rm}^{\mathrm{m}} + RH_v + RH_b)(\mathrm{m}O_v + nO_b + \mathrm{s})}$$

- Detailed derivation of equation (37)

$$2\sqrt{\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + n^*O_b + \mathrm{s})} - 2\sqrt{\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + (n^*-1)O_b + \mathrm{s})} \le 0$$

Dividing by 2 and moving the negative part to the right give

$$\sqrt{\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + n^*O_b + \mathrm{s})} \le \sqrt{\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + (n^*-1)O_b + \mathrm{s})}$$

Squaring both sides

$$\left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right) + h_b\left[\frac{D^2}{2\,n^*\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + n^*O_b + \mathrm{s}) \le \left(h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right) + h_b\left[\frac{D^2}{2\,(n^*-1)\,P} + \left(\frac{(P-D)D}{2P}\right)\right]\right)(mO_v + (n^*-1)O_b + \mathrm{s})$$

Expanding the brackets

$$h_{rm}\left(\frac{D^2}{2P}\right)O_v + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)mO_v + \frac{h_bD^2}{2\,n^*\,P}mO_v + h_bmO_v\left(\frac{(P-D)D}{2P}\right) + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)n^*O_b + \frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b + \frac{h_bD^2}{2\;P}O_b + h_bn^*O_b\left(\frac{(P-D)D}{2P}\right) + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)\mathrm{s} + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s} + \frac{h_bD^2}{2\,n^*\,P}\mathrm{s} + h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)$$
$$\le h_{rm}\left(\frac{D^2}{2P}\right)O_v + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)mO_v + \frac{h_bD^2}{2\,(n^*-1)\,P}mO_v + h_bmO_v\left(\frac{(P-D)D}{2P}\right) + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)(n^*-1)O_b + \frac{h_v}{2}\left(\frac{D^2}{P}\right)O_b + \frac{h_bD^2}{2\;P}O_b + h_b(n^*-1)O_b\left(\frac{(P-D)D}{2P}\right) + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)\mathrm{s} + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)\mathrm{s} + \frac{h_bD^2}{2\,(n^*-1)\,P}\mathrm{s} + h_b\mathrm{s}\left(\frac{(P-D)D}{2P}\right)$$

Removing repeated terms from both sides

$$\frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)mO_v + \frac{h_b D^2}{2\,n^*\,P}\mathrm{m}O_v + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)n^* O_b + h_b n^* O_b\left(\frac{(P-D)D}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,n^*}\right)\mathrm{s} + \frac{h_b D^2}{2\,n^*\,P}\mathrm{s}$$
$$\leq \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)\mathrm{m}O_v + \frac{h_b D^2}{2\,(n^*-1)\,P}\mathrm{m}O_v + h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)(n^*-1)O_b$$
$$+ h_b(n^*-1)O_b\left(\frac{(P-D)D}{2P}\right) + \frac{h_v}{2}\left(\frac{D^2}{P\,(n^*-1)}\right)\mathrm{s} + \frac{h_b D^2}{2\,(n^*-1)\,P}\mathrm{s}$$

Sorting out the common terms

$$\frac{1}{\mathrm{n}^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right] + \mathrm{n}^*\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\leq \frac{1}{\mathrm{n}^*-1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)mO_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right]$$
$$+ (\mathrm{n}^*-1)\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$

Putting all terms on one side gives

$$\frac{1}{\mathrm{n}^*}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)mO_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right]$$
$$-\frac{1}{\mathrm{n}^*-1}\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)mO_v + \frac{h_b D^2}{2\;P}\mathrm{m}O_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right]$$
$$+ \mathrm{n}^*\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$-(\mathrm{n}^*-1)\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right] \leq 0$$

Sorting out the common term gives

$$\left(\frac{1}{\mathrm{n}^*} - \frac{1}{\mathrm{n}^*-1}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)mO_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right]$$
$$+ \left(\mathrm{n}^*-(\mathrm{n}^*-1)\right)\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right] \leq 0$$

$\mathrm{n}^*-(\mathrm{n}^*-1)=1$, then

$$\left(\frac{1}{\mathrm{n}^*} - \frac{1}{\mathrm{n}^*-1}\right)\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right] + \left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]$$
$$\leq 0$$

Putting the unknowns on one side gives

$$\left(\frac{1}{\mathrm{n}^*} - \frac{1}{\mathrm{n}^*-1}\right) \leq \frac{-\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b + h_b O_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v + \frac{h_b D^2}{2\;P}mO_v + \frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s} + \frac{h_b D^2}{2\;P}\mathrm{s}\right]}$$

Doing algebraic manipulation to the left side

$$\left(\frac{(\mathrm{n}^*-1)}{\mathrm{n}^*(\mathrm{n}^*-1)}-\frac{\mathrm{n}^*}{\mathrm{n}^*(\mathrm{n}^*-1)}\right) \leq \frac{-\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)mO_v+\frac{h_bD^2}{2\ P}mO_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\ P}\mathrm{s}\right]}$$

$$\frac{-1}{\mathrm{n}^*(\mathrm{n}^*-1)} \leq \frac{-\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v+\frac{h_bD^2}{2\ P}mO_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\ P}\mathrm{s}\right]}$$

Multiply both sides by -1

$$\frac{1}{\mathrm{n}^*(\mathrm{n}^*-1)} \geq \frac{\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v+\frac{h_bD^2}{2\ P}mO_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\ P}\mathrm{s}\right]}$$

Taking the reciprocal

$$\mathrm{n}^*(\mathrm{n}^*-1) \leq \frac{\left[\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{m}O_v+\frac{h_bD^2}{2\ P}mO_v+\frac{h_v}{2}\left(\frac{D^2}{P}\right)\mathrm{s}+\frac{h_bD^2}{2\ P}\mathrm{s}\right]}{\left[h_{rm}\left(\frac{D^2}{2\mathrm{m}P}\right)O_b+\ h_bO_b\left(\frac{(P-D)D}{2P}\right)\right]}$$

Doing algebraic manipulations to the right side gives

$$\mathrm{n}^*(\mathrm{n}^*-1) \leq \frac{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[h_vmO_v+h_bmO_v+h_v\mathrm{s}+h_b\mathrm{s}\right]}{\frac{\mathrm{D}^2}{2\mathrm{P}}\left[\frac{h_{rm}O_b}{\mathrm{m}}+\ h_bO_b\left(\frac{(P-D)}{D}\right)\right]}$$

$$\mathrm{n}^*(\mathrm{n}^*-1) \leq \frac{mO_v(h_v+h_b)+\mathrm{s}(h_v+h_b)}{\left[\frac{h_{rm}O_b}{\mathrm{m}}+\ h_bO_b\left(\frac{(P-D)}{D}\right)\right]}$$

$$\mathrm{n}^*(\mathrm{n}^*-1) \leq \frac{(h_v+h_b)(mO_v+\mathrm{s})}{O_b\left[\frac{h_{rm}}{\mathrm{m}}+\ h_b\left(\frac{(P-D)}{D}\right)\right]}$$

Vita

Osama Alkhatib was born and grew up in Dubai, United Arab Emirates. He graduated from Al Mahmoud secondary school in 2009 and awarded the 10th among all UAE students. He then enrolled in Khalifa University of Science, Technology and Research (KUSTAR), from which he graduated in 2014, with bachelor's degree in Electrical and Electronics Engineering gaining the award of excellent with high honor. After being awarded for the bachelor degree, Engineer Osama Alkhatib joined the American University of Sharjah (AUS) to study master's degree of science (MSc) in Engineering Systems Management (ESM), and he is currently working with Etisalat in the Digital Service Improvement (DSI) department.