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Harvesting and stocking in discrete-time contest competition models with open problems and conjectures

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Abstract. In this survey, we present a class of first and second-order difference equations representing general form of discrete models arising from single-species with contest competition. Then, we consider various harvesting/stocking strategies and discuss their effect on stability, persistence and maximum sustainable yield. The main aim of this work is to give an account of recent results on the subject within a unified framework, then present some open questions and conjectures that deserve further investigation.

1 Introduction

Difference equations of the form $x_{n+1} = x_n f(x_n)$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$ are used in modeling single-species populations with non-overlapping generations that reproduce at discrete time n, where x_n is the population size at the start of the n^{th} breading season, and the function f represents the net reproductive rate per individual [9, 25, 26, 32]. The form $xf(x)$ is used to stress the zero steady state, and the recruitment function f must be chosen to reflect known observations or facts about the modeled species. For instance, to reflect the limited resources available to a given population, f must be decreasing where resources can be food, water, shelter, mates,... etc. Other characteristics of f may reflect competition for resources among individuals within a species. This type of competition is known as the intra-specific competition, and it is a significant factor for the growth of a population. When individuals among a population exploit a common resource, several factors can influence the amount of resource available to an individual. However, we are interested in a general form of models that reflect two types of intra-specific competition, namely contest and scramble competitions [41, 23, 24]. In brief, when individuals among a population compete for resources that are not available to all individuals, then superior or dominant individuals achieve their needs and survive, while others fail to achieve their needs and consequently vanish. Such a competition is known as contest competition [41]. On the other hand, when resources are evenly distributed among the population, individuals have almost equal chances to exploit the resources, and when the resources become scarce, success becomes incomplete. This form of intra-specific competition is known as scramble competition. For more details about contest and scramble competitions as well as some specific examples of the two forms, we refer the interested reader to [7, 23, 24, 41, 8]. Although the borderline between the two forms of competition is not always sharply defined [8], we are interested in quantifying the scenario of a contest competition, which mostly leads to compensatory models [8, 10, 7]. Therefore, it is natural to confine the recruitment function f so as to obey the following assumptions [1, 2].

(A1) $f \in C([0, \infty))$ and f is decreasing;
(A2) $f(0) = b > 1$;
(A3) $xf(x)$ is increasing and bounded by a constant M.

As we proceed with our discussion and analysis, other smoothness assumptions on f can be added as necessary for the sake of developing a mathematical theory.

When time lag occurs between spawning and recruitment, for instance due to substantial maturation time to sexual maturity, the model must include a delay effect [21]. Thus, it is logical to replace $x_{n+1} = x_n f(x_n)$ with the more general equation

$$x_{n+1} = x_n f(x_{n-k}), \tag{1.1}$$

where k is a nonnegative integer. For instance, the baleen whale model $y_{n+1} = \alpha y_n + f(y_{n-k})$ due to Clark [11] can be transformed to the form in Eq. (1.1). It is known that a delay can have a negative impact on populations by causing oscillations and destabilizing steady states. See [27, 29, 40] for more details. Here, we limit our attention to the case $k \leq 1$ in Eq. (1.1).

When a species goes through controlled or uncontrolled exploitation due to hunting, fishing, emigration or immigration, it is necessary to accommodate these factors by modifying Eq. (1.1), and therefore, it is natural to subtract (or add) a harvesting (or stocking) term. Thus, Eq. (1.1) becomes

$$x_{n+1} = x_n f(x_{n-k}) \pm H_n(x_n, x_{n-1}), \quad k = 0 \quad \text{or} \quad 1, \tag{1.2}$$

where the exact form or character of H_n has to reflect the nature of harvesting/stocking. Harvesting and stocking strategies can be used to prevent species extinctions, to improve the total yield over time, or to force coexistence between different species. AlSharawi and Rhouma [4, 6] considered the discrete Beverton-Holt model in a deterministic environment and investigated the effect of various harvesting strategies. They found that constant harvest is more beneficial to both the population and the maximum sustainable yield (MSY) when the size of the population is sufficiently large, while periodic harvesting has a short-term advantage when the size of the population is low. On the other hand, conditional harvesting (harvesting when the size of population is higher than a certain level and stopping otherwise) has the advantage of lowering the risk of depletion or extinction. Also, AlSharawi and Rhouma used various harvesting/stocking strategies in [5] to show that it is possible to guarantee the survival of the weaker species in a competitive exclusion environment. In other studies [28, 33, 34, 37, 38, 39], constant stocking is found to have the effect of suppressing chaos, reversing the period doubling phenomena, lowering the risk of extension and stabilizing the population steady state.

In the following two sections, we give an account of recent results on difference equations of the form (1.2) modeling single species populations under the effect of various stocking/harvesting strategies. Several open problems and conjectures that deserve further investigation are given throughout the paper.

2 Constant yield harvesting/stocking

In this section, we discuss the dynamics of Eq.(1.2) when $H_n(x_n, x_{n-1})$ is a constant, say h. When $\pm h$ is taken negative, the strategy is known as constant catch or constant yield harvesting [12, 35, 16]. On the other hand, the constant is taken positive when the species is affected by stocking due to, for example, refuge or immigration [28, 39].

2.1 No delay in recruitment ($k = 0$)

Consider the difference equation

$$x_{n+1} = x_n f(x_n) + h, \quad h \in \mathbb{R}. \tag{2.1}$$

At $h = 0$, we have the two equilibrium solutions $\bar{x}_{1,0} = 0$ and $\bar{x}_{2,0} = f^{-1}(1)$. When h is positive (stocking), $\bar{x}_{1,h}$ shifts below zero while $\bar{x}_{2,h}$ shifts above $f^{-1}(1)$. Thus, $\bar{x}_{1,h}$ is beyond our interest and we are left with the positive equilibrium $\bar{x}_{2,h}$, which is increasing in h. Using a simple cobweb diagram, we observe that $\bar{x}_{2,h}$ is globally attractive. On the other hand, if the constant term is taken negative due to harvesting, i.e., $-h$ is taken in place of $+h$, then $\bar{x}_{1,h}$ shifts upward and $\bar{x}_{2,h}$ shifts downward till they collide at a maximum harvesting level

$$h_{\max} := x(f(x) - 1), \quad 0 \leq x \leq f^{-1}(1). \tag{2.2}$$

This level of harvesting gives what is known as the maximum sustainable level. A harvesting level beyond $h_{\max}$ leads to a total collapse of the population, while a harvesting level below $h_{\max}$ assures the survival of all initial populations that are larger than or equal to the small equilibrium $\bar{x}_{1,h}$. Again, a cobweb diagram can be used to show that $(\bar{x}_{1,h}, \infty)$ is the basin of attraction of $\bar{x}_{2,h}$. Thus, the dynamics of Eq. (2.1) is easily characterized; however, further illustrations and discussion can be found in [4, 13].

2.2 One-unit time delay in recruitment ($k = 1$)

Consider the difference equation

$$x_{n+1} = x_n f(x_{n-1}) \pm h, \tag{2.3}$$

where h is a positive parameter representing a constant stocking or harvesting quota. Eq.(2.3) has the same equilibrium solutions as in the case $k = 0$; however, the dynamics becomes a bit more challenging. At $h = 0$, the delay in the recruitment function does not change the boundedness character of solutions; however, in this case, monotonic convergence changes into oscillatory convergence. A solution is called oscillatory about an equilibrium $\bar{x}$ if it does not stay on one side of $\bar{x}$ indefinitely. In a solution $\{x_n\}$, a full consecutive segment of terms above or equal to $\bar{x}$ defines the so-called positive semi-cycle. On the other hand, a full consecutive segment of terms below $\bar{x}$ defines a negative semi-cycle. One way to establish the oscillatory nature of solutions of Eq. (2.3) is by setting a new coordinate system at the positive equilibrium $\bar{x}$, then use the map

$$T_h : \mathbb{R}^{+2} \to \mathbb{R}^{+2} \qquad \text{defined by} \qquad T_h(x, y) = (y, yf(x) \pm h) \tag{2.4}$$

to show that T_h rotates the quadrants of the new coordinate system [1].

Next, we proceed by taking stocking and harvesting as separate cases.

Stocking

Here Eq.(2.3) becomes

$$x_{n+1} = x_n f(x_{n-1}) + h, \quad h > 0. \tag{2.5}$$

Solutions of Eq.(2.5) are bounded as we can see from the fact that $x_{n+1} \geq 0$ and

$$x_{n+2} = x_n f(x_n) f(x_{n-1}) + hf(x_n) + h \leq Mb + hb + h,$$

where b is given in Assumption (A2) and M is the bound given in Assumption (A3). The map T_h defined by $T_h(x, y) = (y, yf(x) + h)$ can be used to portray solutions of Eq.(2.5) as orbits in the positive quadrant. A region R_h is called invariant for Eq.(2.5) if $T_h(R_h) \subseteq R_h$. It was shown in [2] that a bounded and invariant region can be obtained by connecting the points $(0, 0), (0, h), (c_h, bc_h + h), (bc_h + h, bc_h + h), (bc_h + h, 0)$ and $(0, 0)$, respectively with line segments, where c_h is taken to be $\frac{1}{b}\sup_t (bt + h) f(t)$. Since solutions are positive and bounded away from zero, we denote

$$I = \liminf\{x_n\} \quad \text{and} \quad S = \limsup\{x_n\}.$$

From the second iterate $x_{n+2} = x_n f(x_n) f(x_{n-1}) + hf(x_n) + h$, we obtain

$$S \leq Sf(S)f(I) + hf(I) + h \quad \text{and} \quad I \geq If(I)f(S) + hf(S) + h.$$

Now, multiply the first inequality by I and the second by S to obtain $Sf(S) + S \leq If(I) + I$. Thus, $I = S$, and consequently, the positive equilibrium $\bar{x}_{2,h}$ is globally attractive. This approach to prove global attractivity was used by Nyerges in [30]. Another approach can be extracted from [22]. Thus, we have the following result.

Theorem 2.1. *Every solution of Eq.(2.5) is oscillatory about $\bar{x}_{2,h}$ and satisfies*

$$\lim_{n\to\infty} x_n = \bar{x}_{2,h}.$$

Alternatively, since $F(x,y)=yf(x)+h$ maps $[0,\infty)\times[0,\infty)$ into $(0,\infty)$, the semi-cycle approach can be used to prove the global stability as shown by Kocic and Ladas in [18] (Theorem 2.1.1, page 27).

Harvesting

Here we consider the difference equation

$$x_{n+1}=F(x_n,x_{n-1})=x_nf(x_{n-1})-h, \tag{2.6}$$

where $h>0$ represents a harvesting quota. This equation was considered by AlSharawi et al. in [1]. A prototype example of Eq. (2.6) is the well-known Pielou's equation with constant effort harvesting. The dynamics of Eq. (2.6) turns out to be interesting and challenging. Before we proceed further, we give the definitions of persistence and strong persistence. A solution of Eq.(1.2) is called persistent if the corresponding population survives indefinitely. We call a persistent solution strongly persistent if $\liminf x_n>0$. A set $\mathcal{D}:=\{(x,y):\ (x,y)\in{\mathbb{R}^+}^2\}$ is persistent if each solution of Eq.(1.2) with $(x_{-1},x_0)\in\mathcal{D}$ is persistent. In Eq.(2.6), we use $\mathcal{D}_h$ to denote the largest persisting set at the harvesting level h. From Eq.(2.6), persistence implies $x_n\geq\frac{h}{f(x_{n-1})}$, and consequently $x_n\geq\frac{h}{b}$. Thus, persistence implies strong persistence. The next result is obtained from [1] and gives an attempt to characterize the set $\mathcal{D}_h$.

Theorem 2.2. *Consider Eq.(2.6) and define h_{max} as given in Eq.(2.2). Each of the following holds true.*

(i) Persistence implies strong persistence.

(ii) Persistent solutions are bounded.

(iii) If $h>h_{max}$, then $\mathcal{D}_h$ is empty.

(iv) If $h=h_{max}$, then all elements of $\mathcal{D}_h$ are attracted to $\bar{x}_{h_{max}}:=\bar{x}_{1,h}=\bar{x}_{2,h}$.

Computer simulations show that $\mathcal{D}_h$ shrinks as h increases, which is in accord with the fact provided about the basin of attraction of $\bar{x}_{2,h}$ in the absence of time lag; however, a mathematical proof is missing in case of Eq.(2.6). We formalize this observation in the following conjecture.

Conjecture 2.3. Consider Eq.(2.6). The set $\mathcal{D}_h$ is decreasing in h, i.e., if $h_1\leq h_2$ then $\mathcal{D}_{h_2}\subseteq\mathcal{D}_{h_1}$.

The equilibrium $\bar{x}_{1,h}$ is a saddle for all $0\leq h\leq h_{max}$. At $h=0$, the stable manifold does not appear in the positive quadrant, and therefore, $\bar{x}_{1,h}$ can be ignored. However, when $h>0$, the stable manifold of $\bar{x}_{1,h}$ becomes in the persistent set, which spices up the dynamics of Eq.(2.6). A comparison principle was developed and used by AlSharawi et al. in [1] to show that persistent solutions are eventually larger than or equal to $\bar{x}_{1,h}$. The following three results are obtained from [1].

Theorem 2.4. *Let $\{x_n\}$ be a solution of Eq.(2.6) and suppose there are two sequences α_n and β_n such that $\alpha_n\leq x_n\leq\beta_n$ for all $n\geq-1$. Define $w_{n+1}(x)=F(x,\alpha_n)$ and $g_{n+1}(x)=F(x,\beta_n)$, then we obtain*

$$g_ng_{n-1}\cdots g_0(x_0)\leq x_{n+1}\leq w_nw_{n-1}\cdots w_0(x_0).$$

Theorem 2.5. *Consider the initial condition $t_0:=\frac{h}{b}$ in the first-order difference equation $t_{n+1}=\frac{t_n+h}{f(t_n)}$, then t_n converges monotonically to $\bar{x}_{1,h}$. Furthermore, any persistent solution $\{x_n\}_{-1}^{\infty}$ of Eq.(2.6) satisfies $x_n\geq t_n$ for all $n=0,1,\ldots$.*

Theorem 2.6. *A persistent solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.6) satisfies*

$$\bar{x}_{1,h}\leq\liminf x_n\leq\limsup x_n\leq x^*_{2,h},$$

where $x^*_{2,h}$ is the largest fixed point of the function

$$g(t)=h\frac{f(t)+1}{f(t)f(\bar{x}_{1,h})-1}\quad\text{in the interval}\quad\left[0,f^{-1}\left(1/f(\bar{x}_{1,h})\right)\right).$$

Next, we consider a specific case of Eq.(2.6). The authors in [1] forced harvesting on Pielou's equation [31, 20] to obtain

$$y_{n+1} = \frac{Kby_n}{K+(b-1)y_{n-1}} - h^*, \quad b>1, K>0, h^*>0. \tag{2.7}$$

Let $x_{n-1} := \frac{b-1}{K}y_{n-1}$ and $h := \frac{b-1}{K}h^*$. We obtain

$$x_{n+1} = \frac{bx_n}{1+x_{n-1}} - h, \quad b>1, h>0. \tag{2.8}$$

Thus $f(t) = \frac{b}{1+t}$ in Eq.(2.6). We extract the following facts from [1] and provide some questions that worth further investigation. At $h=1$ and $b \geq 4$, Eq.(2.8) is related to Lyness equation [14, 15, 19] and has the invariants

$$\mathcal{I}_b(x_n, x_{n-1}) := \left(1+\frac{b}{1+x_n}\right)\left(1+\frac{b}{1+x_{n-1}}\right)(1+x_n+x_{n-1}) = \mathcal{I}_b(x_0, x_{-1}). \tag{2.9}$$

In this case, the persistence set $\mathcal{D}_1$ can be found explicitly. Indeed, $(x,y) \in \mathcal{D}_1$ if and only if

$$2+(b+1)^2 - \bar{x}_2(b-4) \leq \mathcal{I}_b(x,y) \leq 2+(b+1)^2 - \bar{x}_1(b-4).$$

When $h=1$ and $b > 2(1+\sqrt{2})$, an 8-periodic solution of Eq.(2.8) was found and used to define a trapping region for Eq.(2.8) with $0<h<1$. A subset of the persistent set $\mathcal{D}_h$ is called a trapping region if it is invariant and all persistent solutions enter the region in finite time. We close this section by giving a conjecture and two open problems.

Conjecture 2.7. Consider Eq.(2.8) with $0<h<1$. All persistent solutions larger than $\bar{x}_{1,h}$ are attracted to $\bar{x}_{2,h}$.

Open Problem 2.1. Consider Eq.(2.8) with $0<h<1$. Show that $\mathcal{D}_1 \subseteq \mathcal{D}_h$.

Open Problem 2.2. Consider Eq.(2.8) with $h>1$. Characterize $\mathcal{D}_h$. Is $\mathcal{D}_h$ closed? Is $\mathcal{D}_h$ connected?

3 Periodic Harvesting/Stocking

Harvesting or stocking can be controlled or regulated to prevent species extinction or to improve the total yield over a period of time. However, the question on how to regulate harvesting/stocking is widely open for research and debate [4, 17, 12, 16, 36, 42]. For instance, De Klerk and Gatto considered a continuous multi-cohort Beverton-Holt model in [17] and argued that adopting a periodic fishing strategy instead of a constant effort strategy is worthwhile when there is a significant economy of scale, and when older fish are much more valuable than younger ones. Another interesting example is given by AlSharawi and Rhouma in [4], in which the discrete Beverton-Holt model $x_{n+1} = \frac{bx_n}{1+x_n}$ was investigated under the effect of several harvesting strategies. We quote Figure 1, which summarizes the conclusion of their work.

In this section, we consider periodic stocking/harvesting in Eq.(1.2) and discuss the dynamics for both $k=0$ and $k=1$. We stress here that the period of a periodic sequence is always meant to be minimal. Before we proceed, it is worth mentioning that population cycles evolve under periodic stocking/harvesting and become multiples of the stocking/harvesting period. This result was provided in [2] and we formalize it in the following theorem.

Theorem 3.1. *[2] If there exists an r-periodic solution of the p-periodic difference equation $x_{n+1} = F(x_n, x_{n-1}) \pm h_n$, then r is a multiple of p.*

3.1 No time lag ($k=0$)

Consider the p-periodic equation

$$x_{n+1} = x_n f(x_n) \pm h_n, \;\; h_n \geq 0, \tag{3.1}$$

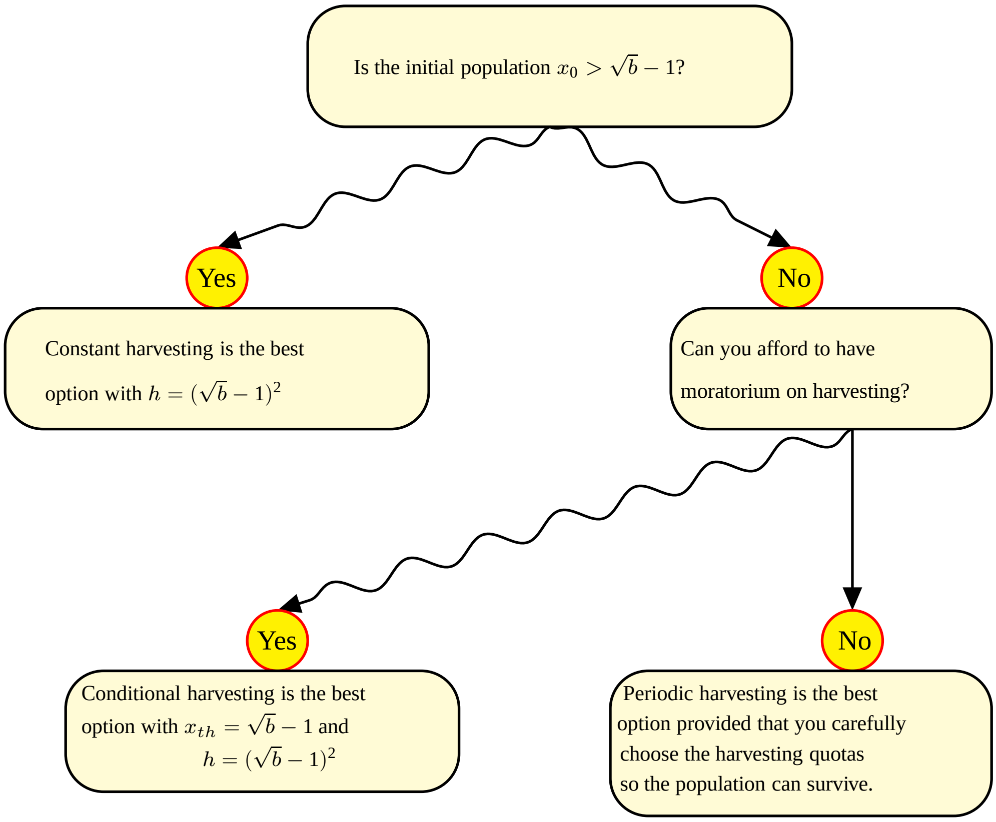

Figure 1. This diagram summarizes the conclusion of the results obtain by AlSharawi and Rhouma in [4] for various harvesting strategies applied to the Beveton-Holt model.

where $\{h_n\}$ is a p-periodic sequence representing periodic stocking or harvesting. Although it is possible not to have stocking in some seasons ($h_j = 0$ for some j), we want to avoid reducing Eq.(3.1) to $x_{n+1} = x_n f(x_n)$, and therefore, we assume $\sum h_n > 0$.

Periodic Stocking

In this case, Eq.(3.1) becomes

$$x_{n+1} = x_n f(x_n) + h_n, \tag{3.2}$$

where $h_n \geq 0$ and $h_{n+p} = h_n$ for all $n \in \mathbb{N}$. Define the maps $f_n(x) = xf(x) + h_n$. Since each map f_n is an upward shift of $y = xf(x)$, the period of any periodic solution has to be a multiple of p [2], also follows from Theorem 3.1. Define the p-fold functions $G_j := f_{p+j-1} \circ f_{p+j-2} \circ \cdots f_j$, then for each $j = 0, 1, \ldots, p-1$, G_j is increasing and bounded with $G_j(0) > 0$. Thus, G_j has a unique positive fixed point, say $\bar{x}_{j,h_n}$. Furthermore,

$$\lim_{n\to\infty} x_{np+j} = \bar{x}_{j,h_n}.$$

Now, $\{\bar{x}_{0,h_n}, \bar{x}_{1,h_n}, \cdots, \bar{x}_{p-1,h_n}\}$ is a p-periodic solution of Eq.(3.2) which is a global attractor. Let $\{h_n\}$ be a p-periodic sequence of stocking quotas. Define

$$h_{av} := \frac{1}{p}\sum_{j=0}^{p-1} h_j.$$

Now, sum Eq.(3.2) over the periodic attractor to obtain

$$\sum_{j=0}^{p-1} \bar{x}_j = \sum_{j=0}^{p-1} \left(\bar{x}_j f(\bar{x}_j) + h_j\right).$$

If $y = tf(t)$ is concave, then we can use Jensen's inequality to conclude that

$$\bar{x}_{av} \leq \bar{x}_{av} f(\bar{x}_{av}) + h_{av}.$$

Thus, $\bar{x}_{av} \leq \bar{x}_{2,h_{av}}$, where $\bar{x}_{2,h_{av}}$ is the globally stable equilibrium at a constant stocking level $h = h_{av}$. This phenomenon is known as attenuance, and in this case, we say populations attenuate under periodic stocking. However, a more ambiguous notion that needs deep investigation is the following. How does the order of the stocking quotas affect the population average? Before we formulate this question into an open problem, we consider an illustrative example. Consider the Beverton-Holt model with p-periodic stocking

$$x_{n+1} = \frac{K\mu x_n}{K + (\mu - 1)x_n} + h_n,\ K > 0, \mu > 1, h_n \geq 0 \text{ and } \sum h_n > 0. \tag{3.3}$$

This equation has a globally asymptotically stable periodic solution of period p, which can be written explicitly. If $p = 2$, then a rearrangement of h_0 and h_1 does not affect the periodic solution. However, as clarified above, populations attenuate. To be more specific, fix $\mu = 2,\ K = 3,\ h_0 = 0$ and $h_1 = h$. Then, it is a simple algebraic computation to find the globally asymptotically stable equilibrium of the equation

$$x_{n+1} = \frac{6x_n}{3 + x_n} + h_{av} = \frac{6x_n}{3 + x_n} + \frac{1}{2}h.$$

Indeed, it is given by

$$\bar{x}_{2,h_{av}} := \frac{3}{2} + \frac{1}{4}h + \frac{1}{4}\sqrt{h^2 + 36h + 36}.$$

On the other hand, the globally asymptotically stable periodic solution of the 2-periodic equation

$$x_{n+1} = \frac{6x_n}{3 + x_n} + h_{n \bmod 2}$$

is given by

$$\{\bar{x}_1, \bar{x}_2\} := \left\{ \frac{1}{2}(3 + h + \sqrt{h^2 + 10h + 9}), \frac{(27 + 3h + 9\sqrt{h^2 + 10h + 9})}{2(h + 9)} \right\},$$

which has an average smaller than or equal to $\bar{x}_{2,h_{av}}$. Next, we proceed to illustrate the rearrangements problem. Consider $p = 3$, then the rearrangements of the stocking quotas are $[h_0, h_1, h_2], [h_0, h_2, h_1]$ and their rotations. Since rotations do not change periodic solutions [3], then we need to compare the averages of the global attractor obtained by taking $\{h_n\} = [h_0, h_1, h_2]$ or $[h_0, h_2, h_1]$. For instance, consider $\mu = 2, K = 3, h_0 = 1, h_1 = 2$ and $h_2 = 2 + h$, then the sequence $\{h_n\} = [1, 2, 2 + h]$ gives a global attractor with an average say $\bar{X}_{av1}$. On the other hand, the sequence $\{h_n\} = [1, 2 + h, 2]$ gives a global attractor with an average say $\bar{X}_{av2}$. Now, it is a computational matter to find that

$$\bar{X}_{av1} = \bar{X}_{av2} + \frac{27h(h + 1)}{46(11h + 103)(5h + 46)} > \bar{X}_{av2}.$$

Now, it is logical to pose the next open problem.

Open Problem 3.1. Let $\{h_n\}$ be a p-periodic sequence of stocking quotas, and let $\{\hat{h}_n\}$ be a permutation of $\{h_n\}$. Define $\bar{x}_{av}$ and $\hat{x}_{av}$ to be the average of the global attractors associated with $\{h_n\}$ and $\{\hat{h}_n\}$, respectively. How does $\bar{x}_{av}$ relate to $\hat{x}_{av}$?

Periodic Harvesting

In this case, Eq.(3.1) becomes

$$x_{n+1} = x_n f(x_n) - h_n, \tag{3.4}$$

where $h_n \geq 0$, $\sum h_j > 0$ and $h_{n+p} = h_n$ for all $n \in \mathbb{N}$. Obviously, sufficiently large values of h_n lead to a total collapse of the population. Thus, finding a maximum sustainable yield

(MSY) is an issue of particular interest here. Define the maps $f_n(x) = xf(x) - h_n$ and $G_j := f_{p+j-1} \circ f_{p+j-2} \circ \cdots f_j$. Now, the MSY can be found from the following constraints

$$G_j(x) = x \quad \text{and} \quad G_j'(x) = 1, \quad \text{for all} \quad j = 0, 1, \ldots, p-1. \tag{3.5}$$

We use the Beverton-Holt model to illustrate the above results, see [9].

Example 3.2. Consider the Beverton-Holt model with 2-periodic harvesting given by

$$x_{n+1} = \frac{K\mu x_n}{K + (\mu - 1)x_n} - h_n = f_n(x_n), \tag{3.6}$$

where $h_{n+2} = h_n$ for all $n \in \mathbb{N}$ and $h_0, h_1 > 0$. Based on the constraints in Eqs.(3.5), we can eliminate x and obtain a relationship between h_0 and h_1. Indeed, we obtain

$$K^2 - K\beta(h_0 + h_1) + h_0 h_1 = 0, \quad \beta = \frac{\mu + 1}{\mu - 1},$$

or equivalently

$$h_1 = \frac{K(K - \beta h_0)}{K\beta - h_0}, \quad h_0 < \frac{K}{\beta}.$$

We use the relationship between h_0 and h_1 to find

$$\bar{x}_0 = \frac{K(K + h_0)}{K(\mu + 1) - h_0(\mu - 1)} \quad \text{and} \quad \bar{x}_1 = \frac{1}{2}(K - h_0).$$

At $h_0 = \frac{(\sqrt{\mu}-1)}{\sqrt{\mu}+1}K$, we obtain $h_0 = h_1$ and $\bar{x}_0 = \bar{x}_1$, which is the constant harvesting case. Observe that a swap of h_0 and h_1 leads to a swap of $\bar{x}_0$ and $\bar{x}_1$, which seems to be of little mathematical effect, but in fact, it has a dramatic effect when the size of the population is low. When $h_0 < h_1$, we have $\bar{x}_0 < \bar{x}_1$ and populations in $[\bar{x}_0, \infty)$ persist. On the other hand, $h_0 > h_1$ implies $\bar{x}_0 > \bar{x}_1$ and $[\bar{x}_0, \infty)$ is the persistent set. Therefore, one can investigate the advantage of having $0 \leq h_0 \leq h_1 \leq h_{max}$ at all times. See Figure 2 for an illustration.

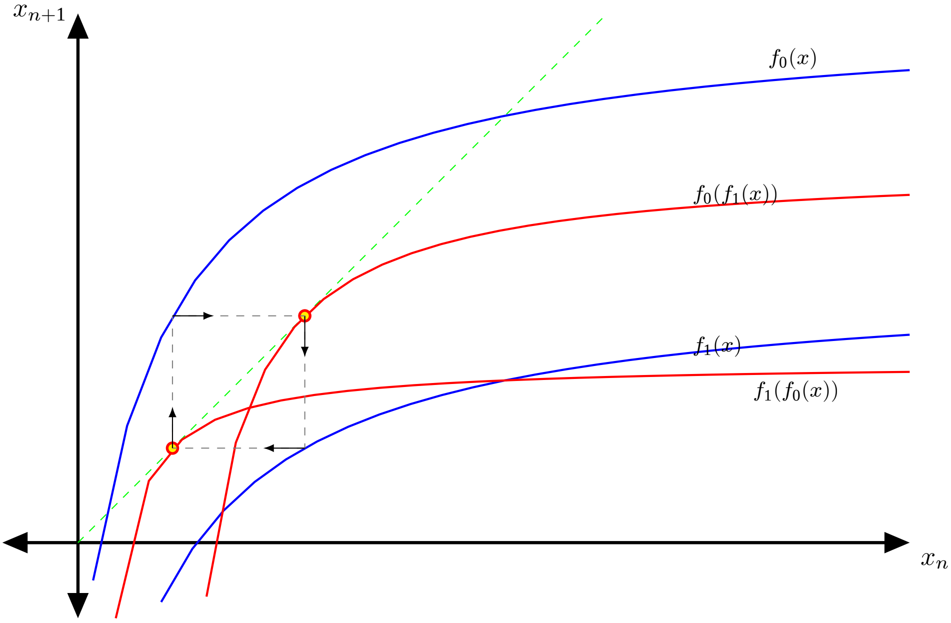

Figure 2. This figure shows the curves of $f_0(x), f_1(x), f_1(f_0(x))$ and $f_0(f_1(x))$ together with the 2-cycle $\{\bar{x}_0, \bar{x}_1\}$. The parameters are fixed as $K = 4, \mu = 9, h_0 = 1$ and $h_1 = \frac{11}{4}$.

Results presented in Example 3.2 motivate investigating the following open problems. We use $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$ to denote the persistent set of Eq.(3.6).

Open Problem 3.2. Let $\{h_n\}$ be a p-periodic sequence of harvesting quotas in Eq.(3.6) that give a nonempty persistent set $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$. Let $\{\hat{h}_n\}$ be a permutation of $\{h_n\}$. Define $\overline{x}_{av}$ and $\hat{x}_{av}$ to be the average of the attractors associated with $\{h_n\}$ and $\{\hat{h}_n\}$, respectively. How does $\overline{x}_{av}$ relate to $\hat{x}_{av}$?

Open Problem 3.3. In Eq.(3.6), let $\{h_n\}$ be a fixed p-periodic sequence of harvesting quotas that give a nonempty persistent set $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$. Which permutation of $\{h_n\}$ gives the largest persistent set?

Open Problem 3.4. What happens to the invariants given in Eq.(2.9) when $h_n = 1 \pm \epsilon_n$?

In conjunction with these three open questions, it is worth mentioning that AlSharawi and Rhouma investigated in [5] the effect of permuting a periodic carrying capacity on the maximum sustainable yield. In particular, they considered the equation

$$x_{n+1} = \frac{k_{j_n} \mu x_n}{k_{j_n} + (\mu - 1)x_n} - h, \quad n \in \mathbb{N}, \tag{3.7}$$

where $(j_0, j_1, \ldots, j_{p-1})$ is a permutation of the set $\{0, 1, 2, \ldots, p-1\}$ and $k_{j_{n+p}} = k_{j_n}$ for all positive integers n, and obtained the following two results:

Theorem 3.3. *[5] Fix a p-periodic sequence of carrying capacities $[k_0, k_1, \ldots, k_{p-1}]$. All equations of the form (3.7) with permutations $(j_0, j_1, \ldots, j_{p-1})$ in the dihedral group of order p give the same maximum constant harvesting level.*

Theorem 3.4. *[5] Consider Eq. (3.7) and assume the initial population is sufficiently large. Without loss of generality, let $k_0 \leq k_1 \leq \cdots \leq k_{p-1}$. Each of the following holds true:*

(i) For $p = 2$ or 3, a permutation of the carrying capacities does not change the maximum harvesting level.

(ii) For $p = 4$, there are three different levels of maximum harvesting through permutations of the carrying capacities. In particular, $(j_0, j_1, j_2, j_3) = (0, 2, 1, 3)$ or $(3, 1, 2, 0)$ and their cyclic permutations give the largest, and $(j_0, j_1, j_2, j_3) = (3, 2, 0, 1)$ or $(1, 0, 2, 3)$ and their cyclic permutation give the smallest.

(iii) For $p = 5$, there are twelve different levels of maximum harvesting through permutations of the carrying capacities. In particular, $(j_0, j_1, j_2, j_3, j_4) = (1, 2, 3, 0, 4)$ or $(4, 0, 3, 2, 1)$ and their cyclic permutations give the largest, and $(j_0, j_1, j_2, j_3, j_4) = (3, 1, 0, 2, 4)$ or $(4, 2, 0, 1, 3)$ and their cyclic permutation give the smallest.

3.2 One-unit time lag ($k = 1$)

Consider the p-periodic second-order difference equation

$$x_{n+1} = x_n f(x_{n-1}) \pm h_n, \tag{3.8}$$

where p is the minimal positive integer for which $h_{n+p} = h_n$ for all n. By considering periodic stocking or harvesting in addition to the delay, we add another factor of complexity to the equation.

Periodic Stocking

In this case we have

$$x_{n+1} = x_n f(x_{n-1}) + h_n, \tag{3.9}$$

where $h_n \geq 0$ is a p-periodic sequence representing stocking quotas ($\sum h_j > 0$). Eq. (3.9) was investigated by AlSharawi in [2]. To capture the main results in [2], we need to cite part of the developed machinery. Define the two dimensional maps $T_j(x, y) = (y, yf(x) + h_j)$, then the iterates of the p-periodic sequence of maps $T_j : \ j = 0, 1, \ldots, p-1$ portray the dynamics of Eq.(3.9) in the positive quadrant. It is possible to define a compact region R_{h_j} that serves as a compact invariant for each individual map T_j. However, we need a compact invariant for the p-fold map $T = T_{p-1} \circ T_{p-2} \circ \cdots \circ T_0$. It was shown that $h_i \leq h_j$ implies $R_{h_i} \subseteq R_{h_j}$, which suggests defining one invariant for all maps T_j. Indeed, consider $h_m := \max_j \{h_j, j = 0, 1, \ldots, p-1\}$ and

$$c_m := \max_j \left\{ \frac{1}{b} \sup_{t \geq 0} (bt + h_j) f(t) : \ j = 0, 1, \ldots, p-1 \right\},$$

then use h_m and c_m to define a region R_{h_m} as the polygon of consecutive vertices $(0,0)$, $(0, h_m)$, $(c_m, bc_m + h_m)$, $(bc_m + h_m, bc_m + h_m)$, $(bc_m + h_m, 0)$. The region R_{h_m} is a compact invariant for each map T_j, and consequently it is a compact invariant for the p-fold map T. Using Brouwer Fixed-Point Theorem, T has a fixed point in R_{h_m}. Based on Theorem 3.1, Eq.(3.9) has a p-periodic solution. These facts furnish the ground for the following result, which is the main result obtained in [2].

Theorem 3.5. *The p-periodic difference equation in Eq.(3.9) has a p-periodic solution. Furthermore, the p-periodic solution is globally attracting when $p = 2$.*

Conjecture 3.6. The p-periodic solution of Eq.(3.9) obtained in Theorem 3.5 is globally attracting for all $p > 2$.

Finally, after verifying this conjecture, then Problem 3.1 can be investigated for Eq.(3.9).

Periodic Harvesting

In this case, Eq.(3.8) takes the form

$$x_{n+1} = x_n f(x_{n-1}) - h_n, \quad h_n \geq 0. \tag{3.10}$$

To the best of our knowledge, Eq.(3.10) has not been studied yet. Define

$$h_{min} = \min\{h_0, h_1, \ldots, h_{p-1}\},$$

then

$$x_{n+2} = x_n f(x_n) f(x_{n-1}) - h_n f(x_n) - h_{n+1} \leq Mb - h_{min}(b+1).$$

Since persistent solutions must satisfy $x_n \geq \frac{h_{min}}{b}$, persistent solutions are bounded and strongly persistent, which is similar to the result in Theorem 2.2. Furthermore, we give the following.

Proposition 3.7. *Consider h_{max} as defined in Eq.(2.2). If $h_n > h_{max}$ for all $n = 0, 1, \ldots, p-1$, then the persistent set of Eq.(3.10) is empty.*

Proof. Let $\{x_n\}$ be a persistent solution such that $x_{-1} \geq x_0$. We obtain $x_1 = x_0 f(x_{-1}) - h_0 < x_0 f(x_0) - h_0 < x_0$. By induction, we obtain a decreasing and bounded sequence that must converge to a value, say α. For $j = 0, 1, \ldots, p-1$, we have

$$x_{np+j} = x_{np+j-1} f(x_{np+j-2}) - h_{j-1},$$

which implies α is a fixed point of this equation. Since $h_{j-1} > h_{\max}$, we obtain a contradiction. Next, start with (x_{-1}, x_0) such that $x_{-1} < x_0$, then either we obtain a monotonic sequence which leads us to a contradiction, or we obtain $x_{m-1} \geq x_m$ for some fixed m and we use the first case to obtain a contradiction. □

Now, we close this section by posing the following open problem.

Open Problem 3.5. Investigate the dynamics of Eq.(3.10).

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