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Commuting Toeplitz Operators With Mixed Quasihomogeneous and Analytic Symbols

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ABSTRACT

A major open problem in the theory of Toeplitz operators on the analytic Bergman space over the unit disk is the characterization of the commutant of a given Toeplitz operator, that is, the set of all bounded Toeplitz operators that commute with it. In this paper, we provide a complete description of bounded Toeplitz operators T_f, where the symbol f has a truncated polar decomposition, that commute with a Toeplitz operator, whose symbol is the sum of a quasihomogeneous function and a bounded analytic function.

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1 | Introduction

In the complex plane $\mathbb{C}$, let $\mathbb{D}$ denote the open unit disk, and we consider the normalized Lebesgue measure $dA = \mathrm{rdr}(d\theta)/\pi$ on $\mathbb{D}$, where (r, θ) are the polar coordinates. The space $L^2(\mathbb{D}, dA)$ consists of all square-integrable functions on $\mathbb{D}$ with respect to this measure.

The classical unweighted Bergman space, denoted by $L^2_a(\mathbb{D})$, is the closed subspace of $L^2(\mathbb{D}, dA)$ comprising all functions that are analytic in $\mathbb{D}$. The set $\{z^k\colon k = 0, 1, 2, \ldots\}$ forms an orthogonal basis for $L^2_a(\mathbb{D})$. Since this space is closed, there exists a well-defined orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$, commonly referred to as the Bergman projection. For a comprehensive treatment of Bergman spaces and their associated projections, readers may refer to [1].

Given a function $f \in L^1(\mathbb{D}, dA)$, the Toeplitz operator T_f acting on $L^2_a(\mathbb{D})$ is defined by $T_f(g) = P(fg)$, provided that $fg \in L^2(\mathbb{D}, dA)$. Here, f is referred to as the symbol of the Toeplitz operator T_f. This definition implies that any bounded analytic function on $\mathbb{D}$ belongs to the domain of T_f, ensuring that T_f is densely defined. Moreover, if the symbol f is bounded on $\mathbb{D}$, the Toeplitz operator T_f is also bounded, with the norm satisfying $\|T_f\| \le \|f\|_\infty$. Nevertheless, an unbounded symbol f belonging to $L^1(\mathbb{D}, dA)$ may still generate a bounded Toeplitz operator T_f. In fact, if f is integrable and remains bounded on the annulus $\{z \in \mathbb{D}\colon 0 < r < |z| < 1\}$ for some $0 < r < 1$, then f can be decomposed into the sum of two functions: one that is integrable with compact support and another that is bounded. Consequently, T_f remains bounded. Such symbols are referred to as "nearly bounded symbols" ([2], p.204). From now on, we will consider any symbol that does not belong to $L^1(\mathbb{D}, dA)$ to be inadmissible for a bounded Toeplitz operator, meaning such a symbol cannot define a bounded Toeplitz operator.

A function f is called quasihomogeneous of degree p, where $p \in \mathbb{Z}$ an integer, if it can be expressed in the form $f(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function. The Toeplitz operator T_f associated with such a symbol is called a quasihomogeneous Toeplitz operator of degree p (see [3, 4]). The study of these operators is motivated by the structural decomposition of $L^2(\mathbb{D}, dA)$, which can be written as $L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}$, where $\mathcal{R}$ denotes the space of square-integrable radial functions on [0,1) with respect to the measure rdr. This decomposition implies that any function $f \in L^2(\mathbb{D}, dA)$ admits a polar expansion $f(z) = f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r)$, where each $f_k(r)$ is a radial function. The subclass of quasihomogeneous Toeplitz operators provides a natural framework for understanding the behavior of operators that respect this decomposition (see [3–8]).

Studying this family of operators offers insight into the commutant of Toeplitz operators associated with more general symbols. By focusing on quasihomogeneous symbols, we aim to uncover structural properties and extend our understanding of operator commutativity in the Bergman space setting.

Over the past several decades, Toeplitz operators have been a central object of study, particularly in terms of their algebraic and functional properties. One of the most intriguing open problems involves characterizing the commutant of a Toeplitz operator, i.e., determining the set of all Toeplitz operators that commute with a given one under composition. Despite substantial progress, little is known about the commutativity of Toeplitz operators with general symbols. For an overview of known results on the commutativity of Toeplitz operators in $L^2_a(\mathbb{D})$ and specific cases where progress has been made, see [3–17].

The commutativity problem for Toeplitz operators on Bergman-type spaces remains a central theme in operator theory, but it is natural to view it within the broader framework of harmonic analysis on non-Euclidean structures and integral geometry. Recent work of De Mariet al. [18] on horocyclic harmonic Bergman spaces over homogeneous trees extends the Bergman space methodology to discrete models of hyperbolic geometry, providing a fertile ground where notions of projection operators, reproducing kernels, and boundedness phenomena echo those in the classical setting. Likewise, Rubin's study [19] of the hemispherical transform and related Radon transforms in the Euclidean half-space highlights deep connections between integral transforms, harmonic analysis, and function spaces arising in geometric contexts. Although the techniques in these works differ from ours, they illustrate the breadth of contexts in which Bergman-type spaces and operator commutativity phenomena arise. These perspectives suggest interesting avenues for future research: one may explore whether commutativity results for Toeplitz operators on the Bergman space can be adapted or extended to harmonic Bergman spaces on trees, or whether techniques from Radon-type transforms can shed new light on the structure of Toeplitz operator algebras. In this way, our present results can be situated within a wider analytic landscape, pointing toward potential interactions between operator theory, discrete models of Bergman spaces, and modern integral geometry. By situating our study within this wider landscape, we aim to emphasize the relevance of our results to ongoing developments in both complex analysis and harmonic analysis.

A preliminary version of this work has been made publicly available as a preprint on arXiv.org [20].

2 | Tools

The Mellin transform $\widehat{\phi}$ of a radial function $\phi \in L^1([0,1), rdr)$ is defined as

$$\widehat{\phi}(z) = \int_0^1 \phi(r) r^{z-1} dr. \tag{1}$$

It is well known that for such functions, the Mellin transform is bounded on the right half-plane $\{z: \Re z \geq 2\}$ and is analytic on $\{z: \Re z > 2\}$.

The following lemma gives the action of quasihomogeneous Toeplitz operators on the vectors of the orthogonal basis of $L^2_a(\mathbb{D})$.

Lemma 1 (see [[4], Lemma 5.3, p.531]). *Let $k, p \in \mathbb{Z}_+$ and let ϕ be a radial function in $L^1([0,1), rdr)$. Then,*

$$T_{e^{ip\theta}\phi}\left(z^k\right) = 2(k+p+1)\widehat{\phi}(2k+p+2) z^{k+p}, \tag{2}$$

and

$$T_{e^{-ip\theta}\phi}\left(z^k\right) = \begin{cases} 0, & \text{if } 0 \leq k \leq p-1, \\ 2(k-p+1)\widehat{\phi}(2k-p+2) z^{k-p}, & \text{if } k \geq p. \end{cases} \tag{3}$$

The Mellin transform of a function is uniquely determined by its values on any arithmetic sequence of integers. In fact, we have the following classical theorem ([21], p.102).

Theorem 1. *Suppose that f is a bounded analytic function on the right half-plane $\{z \in \mathbb{C}: \Re z > 0\}$ that vanishes at a set of pairwise distinct points $d_1, d_2, \ldots$, where*

i. $\inf\{|d_n|\} > 0$.

ii. $\sum_{n\geq 1} \Re(1/d_n) = \infty$.

Then, f vanishes identically on $\{z \in \mathbb{C}: \Re z > 0\}$.

We shall often use the following classical lemma (see also [[15], Lemma 7, p.1727]).

Lemma 2. *If a meromorphic function in the right half-plane belonging to the Nevanlinna class is periodic, then it is constant.*

Remark 1.

i. A direct calculation shows that $\widehat{r^n}(z) = 1/(z+n)$, for $n \in \mathbb{Z}$.

ii. As for Theorem 1, we will apply it in the following setting: let $(n_k)_k$ be an arithmetic sequence of positive integers, and suppose that for a certain radial function ϕ, we have $\widehat{\phi}(n_k) = 0$ for all k. By Theorem 1, this implies that $\widehat{\phi}$ is identically zero on the right half-plane. Consequently, ϕ must also vanish on the right half-plane.

iii. We frequently rely on Lemma 2 in our arguments. In fact, in our proofs, we often encounter equations of the form

$$F(z+p) - F(z) = G(z+p) - G(z), \tag{4}$$

where $\Re(z) > 0$, p is an integer, and F and G are bounded analytic functions on the right half-plane. By applying Lemma 2, we can deduce that $F(z) = c + G(z)$ for some constant c.

3 | Main Theorem

Given a symbol

$$g\left(re^{i\theta}\right) = e^{i\theta}r^3 + \sum_{l=1}^{\infty} a_l \bar{z}^l, \quad a_l \in \mathbb{C}, \text{ where } a_l \neq 0 \text{ for at least one } l \geq 5, \tag{5}$$

we aim to characterize all symbols of the form (i.e., symbols whose polar decomposition is truncated above)

$$f\left(re^{i\theta}\right) = \sum_{n=-\infty}^{N} e^{in\theta} f_n(r), \quad N \geq 1, \tag{6}$$

for which the associated Toeplitz operator T_f commutes with T_g. It is understood here that $f_N \neq 0$. We recall that T_f commutes with T_g if and only if

$$T_f T_g\left(z^k\right) = T_g T_f\left(z^k\right), \tag{7}$$

for all vectors z^k in the orthogonal basis of $L_a^2(\mathbb{D})$. Equivalently, using the notion of commutator of two operators T and S, namely, $[T, S] = TS - ST$, this condition can be expressed as

$$\left[T_f, T_g\right]\left(z^k\right) = T_f T_g\left(z^k\right) - T_g T_f\left(z^k\right) = 0, \quad \text{for all } k \geq 0. \tag{8}$$

Our main theorem can be stated as follows.

Theorem 2. *Let g be a symbol of the form $g(re^{i\theta}) = e^{i\theta}r^3 + \sum_{l=1}^{\infty} a_l \bar{z}^l$, where $z = re^{i\theta}$, $a_l \in \mathbb{C}$ and $a_l \neq 0$ for at least one $l \geq 5$. If there exists a nonzero function f of the form $f(re^{i\theta}) = \sum_{n=-\infty}^{N} e^{in\theta} f_n(r)$, with $N \geq 1$, such that the commutator $[T_f, T_g] = 0$, then T_f is a polynomial of degree at most one in T_g. In other words, there exist constants C_1, C_0 such that $T_f = C_1 T_g + C_0 I$, where I denotes the identity operator.*

It is important to highlight our motivation for choosing the term $e^{i\theta}r^3$ in the symbol g. In the work of Le and Tikaradze ([15], Theorem 3, p.1725), the analytic polynomial part of the symbol is replaced by a quasihomogeneous symbol. While this is not the most general choice, it is also not selected merely to simplify our arguments. Notably, a Toeplitz operator with the symbol $e^{i\theta}r^3$ always has powers, making the characterization of the commutant of T_g significantly more challenging, as we will demonstrate in the final section. We are firmly convinced that our approach extends to more general quasihomogeneous symbols. However, such generalizations would lead to considerably more intricate calculations, which, while feasible, can become cumbersome and lengthy. We hope our choice convinces readers of the significance of our result and the arguments, which differ entirely from those used in the proof of ([15], Theorem 3, p.1725). In fact, the latter relies on techniques that cannot be directly applied to our case, necessitating a completely different approach.

4 | Calculations

Lemma 3. *Under the assumptions of Theorem 2, if equation (8) holds, then*

$$T_{e^{iN\theta}f_N} = C_N\left(T_{e^{i\theta}r^3}\right)^N \text{ and } T_{e^{i(N-1)\theta}f_{N-1}} = C_{N-1}\left(T_{e^{i\theta}r^3}\right)^{N-1}, \tag{9}$$

for some constants C_N and C_{N-1}.

Proof 1. In equation (8), the term z^{k+N+1} comes only from

$$T_{e^{i\theta}r^3} T_{e^{iN\theta}f_N}\left(z^k\right) = T_{e^{iN\theta}f_N} T_{e^{i\theta}r^3}\left(z^k\right), \quad \text{for all } k \geq 0. \tag{10}$$

Thus, $T_{e^{iN\theta}f_N}$ commutes with $T_{e^{i\theta}r^3}$. Hence,

$$\begin{aligned} T_{e^{iN\theta}f_N}\left(z^k\right) &= 2(k+N+1)\widehat{f}_N(2k+N+2)z^{k+N} \\ &= C_N\left(T_{e^{i\theta}r^3}\right)^N\left(z^k\right) \\ &= C_N \prod_{j=0}^{N-1} \frac{2(k+j+2)}{(2k+2j+6)} z^{k+N} \\ &= C_N \frac{2k+4}{2k+2N+4} z^{k+N}, \end{aligned} \tag{11}$$

for some constant C_N. Here, the first equality results from Lemma 1, the second from ([5], Proposition 7, p.1469), and the third from ([5], Lemma 3, p. 1467).

By a similar argument, we prove that $T_{e^{i(N-1)\theta}f_{N-1}} = C_{N-1}(T_{e^{i\theta}r^3})^{N-1}$, for some constant C_{N-1}. □

Lemma 4. *Under the assumptions of Theorem 2, if equation (8) holds, then*

$$\widehat{f_{N-2}}(z+N) = B_{N,N-2}\frac{z+4}{(z+2N-2)(z+2N)} + a_1C_N\frac{z+4}{(z+2N-2)(z+2N)}\sum_{i=0}^{N-1}\frac{z+2i}{z+2i+4}, \tag{12}$$

for some constant $B_{N,N-2}$.

Proof 2. In equation (8), the term z^{k+N-1} comes from

$$\left[T_{e^{i\theta}r^3}, T_{e^{i(N-2)\theta}f_{N-2}}\right]\left(z^k\right) = \left[T_{e^{iN\theta}f_N}, T_{a_1\overline{z}}\right]\left(z^k\right), \quad \text{for } k \geq 0. \tag{13}$$

It is understood here that $a_1 \neq 0$. Thus, using Lemma 1, we obtain that

$$\begin{aligned}(2k+2N-2)\widehat{f_{N-2}}(2k+N)\frac{2k+2N}{2k+2N+2} - \frac{2k+4}{2k+6}(2k+2N)\widehat{f_{N-2}}(2k+N+2)\\ = a_1\frac{2k}{2k+2}(2k+2N)\widehat{f_N}(2k+N) - a_1\frac{2k+2N}{2k+2N+2}(2k+2N+2)\widehat{f_N}(2k+N+2)\\ = a_1C_N\frac{2k(2k+2)}{(2k+2)(2k+2N+2)} - a_1C_N\frac{(2k+2N)(2k+4)}{(2k+2N+2)(2k+2N+4)}.\end{aligned} \tag{14}$$

We complexify the equation above by taking $z = 2k$, and we use Remark 1 (ii) and multiply both sides by $(z+2N+2)/(z+4)$ to obtain

$$\begin{aligned}(z+2N-2)\widehat{f_{N-2}}(z+N)\frac{z+2N}{z+4} - \frac{z+2N+2}{z+6}(z+2N)\widehat{f_{N-2}}(z+N+2)\\ = a_1C_N\frac{z}{z+4} - a_1C_N\frac{z+2N}{z+2N+4}\\ = a_1C_N\sum_{i=0}^{N-1}\frac{z+2i}{z+2i+4} - a_1C_N\sum_{i=1}^{N}\frac{z+2i}{z+2i+4}.\end{aligned} \tag{15}$$

By using Remark 1 (iii), we conclude that there exists a constant $B_{N,N-2}$ such that

$$\widehat{f_{N-2}}(z+N) = B_{N,N-2}\frac{z+4}{(z+2N-2)(z+2N)} + a_1C_N\frac{z+4}{(z+2N-2)(z+2N)}\sum_{i=0}^{N-1}\frac{z+2i}{z+2i+4}. \tag{16}$$

□

Similarly, we state the following lemma, omitting the proof.

Lemma 5.

i. *By considering the term z^{k+N-2} that comes from*

$$\left[T_{e^{i\theta}r^3}, T_{e^{i(N-3)\theta}f_{N-3}}\right]\left(z^k\right) = \left[T_{e^{iN\theta}f_N}, T_{a_2\overline{z}^2}\right]\left(z^k\right) + \left[T_{e^{i(N-1)\theta}f_{N-1}}, T_{a_1\overline{z}}\right]\left(z^k\right), \tag{17}$$

and assuming that a_2, a_1 are both nonzero, we obtain that

$$\begin{aligned}\widehat{f_{N-3}}(z+N-1) = {}& B_{N,N-3}\frac{z+4}{(z+2N-2)(z+2N-4)} \\ &+ \frac{a_2C_N(z+4)}{(z+2N-2)(z+2N-4)}\sum_{i=0}^{N-1}\frac{(z+2i-2)(z+2i)}{(z+2i+2)(z+2i+4)} \\ &+ a_1C_{N-1}\frac{z+4}{(z+2N-2)(z+2N-4)}\sum_{i=0}^{N-2}\frac{(z+2i)}{(z+2i+4)},\end{aligned} \tag{18}$$

for some constant $B_{N,N-3}$.

ii. *By considering the term z^{k+N-3} that comes from*

$$\begin{aligned}\left[T_{e^{i\theta}r^3}, T_{e^{i(N-4)\theta}f_{N-4}}\right]\left(z^k\right) = {}& \left[T_{e^{iN\theta}f_N}, T_{a_3\overline{z}^3}\right]\left(z^k\right) + \left[T_{e^{i(N-1)\theta}f_{N-1}}, T_{a_2\overline{z}^2}\right]\left(z^k\right) \\ &+ \left[T_{e^{i(N-2)\theta}f_{N-2}}, T_{a_1\overline{z}}\right]\left(z^k\right),\end{aligned} \tag{19}$$

and assuming that none of the a_3, a_2, a_1 is zero, we obtain that

$$\begin{aligned}\widehat{f_{N-4}}(z+N-2) = {}& B_{N,N-4}\frac{(z+4)}{(z+2N-6)(z+2N-4)} \\ &+ \frac{a_3C_N(z+4)}{(z+2N-6)(z+2N-4)}\sum_{i=0}^{N-1}\frac{(z+2i-2)(z+2i-4)}{(z+2i+2)(z+2i+4)} \\ &+ \frac{a_2C_{N-1}(z+4)}{(z+2N-6)(z+2N-4)}\sum_{i=0}^{N-2}\frac{(z+2i-2)(z+2i)}{(z+2i+2)(z+2i+4)} \\ &+ \frac{a_1B_{N,N-2}(z+4)}{(z+2N-6)(z+2N-4)}\sum_{i=0}^{N-3}\frac{(z+2i)}{(z+2i+4)} \\ &+ \frac{a_1a_1C_N(z+4)}{(z+2N-6)(z+2N-4)} \\ &\times \sum_{j=0}^{N-2}\sum_{i=0}^{N-2-j}\frac{(z+2i)(z+2i-2+2j)}{(z+2i+4)(z+2i+2+2j)},\end{aligned} \tag{20}$$

for some constant $B_{N,N-4}$.

Remark 2. We draw the reader's attention to the fact that if a_i is zero, then the corresponding symbol $a_i\bar{z}^i$ is zero, and consequently, any commutator involving this term is also zero. Thus, by assuming that these a_i's are nonzero, we are considering the most general case without simplifying the problem.

5 | Proof of the Main Theorem

Lemmas 3, 4, and 5 will guide us through the steps toward proving the main result. In the following three propositions, we will show that, under the assumptions of Theorem 2, if equation (8) holds, then the value N in the symbol f must equal the quasihomogeneous degree of $e^{i\theta}r^3$ in the symbol g, which is 1.

Proposition 1. *Under the assumptions of Theorem 2, if equation (8) holds, then* $N \le 4$.

Proof 3. Suppose $N \ge 5$. By applying partial fraction decomposition to the term on the right-hand side of the expression for $\widehat{f_{N-3}}$ in Part "(i)" of Lemma 5 and using Remark 1 (ii), we see that the radial function f_{N-3} is a polynomial in r and contains the term

$$a_2 C_N \frac{8}{(2N-4)(2N-6)} r^{-N+3}. \tag{21}$$

Since $N \ge 5$, it follows that $-N+3 \le -2$. Consequently, the term in (21) belongs to $L^1([0,1), rdr)$ only if $a_2 C_N = 0$. Given that we assume $a_2 \neq 0$, we must have $C_N = 0$, which contradicts our assumption that $f_N \neq 0$. Therefore, we conclude that $N < 5$, meaning $N \le 4$. □

Proposition 2. *Under the assumptions of Theorem 2, if equation (8) holds, then* $N \le 3$.

Proof 4. Suppose $N = 4$. By taking $k = 0$ in equation (19), we obtain

$$2\widehat{f_0}(2) - 4\widehat{f_0}(4) = -a_3 C_4 \frac{1}{5} - a_2 C_3 \frac{3}{10} - a_1 B_{4,2} \frac{1}{2} - a_1 a_1 C_4 \frac{1}{2} \sum_{i=1}^{3} \frac{2i}{2i+4}. \tag{22}$$

□

On the other hand, from Part "(ii)" of Lemma 5, we have

$$\begin{aligned} 2\widehat{f_0}(2) = {} & B_{4,0} + a_3 C_4 \sum_{i=0}^{3} \frac{(2i-2)(2i-4)}{(2i+2)(2i+4)} + a_2 C_3 \sum_{i=0}^{2} \frac{2i(2i-2)}{(2i+2)(2i+4)} \\ & + a_1 B_{4,2} \sum_{i=0}^{1} \frac{2i}{2i+4} + a_1 a_1 C_4 \sum_{j=0}^{2} \sum_{i=0}^{2-j} \frac{2i(2i-2+2j)}{(2i+4)(2i+2+2j)}, \end{aligned} \tag{23}$$

and

$$4\widehat{f_0}(4) = B_{4,0} + a_3 C_4 \sum_{i=1}^{4} \frac{(2i-2)(2i-4)}{(2i+2)(2i+4)} + a_2 C_3 \sum_{i=1}^{3} \frac{2i(2i-2)}{(2i+2)(2i+4)} + a_1 B_{4,2} \sum_{i=1}^{2} \frac{2i}{2i+4} + a_1 a_1 C_4 \sum_{j=0}^{2} \sum_{i=1}^{3-j} \frac{2i(2i-2+2j)}{(2i+4)(2i+2+2j)}. \tag{24}$$

Thus, the left-hand simplifies to

$$2\widehat{f_0}(2) - 4\widehat{f_0}(4) = a_3 C_4 - a_3 C_4 \frac{1}{5} - a_2 C_3 \frac{3}{10} - a_1 B_{4,2} \frac{1}{2} - a_1 a_1 C_4 \frac{1}{2} \sum_{j=0}^{2} \frac{6-2j}{10-2j}. \tag{25}$$

Equating both sides, we conclude that $a_3C_4 = 0$. Since we are assuming $a_3 \neq 0$, it follows that $C_4 = 0$. Hence, $N \leq 3$.

Proposition 3. *Under the assumptions of Theorem 2, if equation (8) holds, then $N \leq 2$.*

Proof 5. Suppose $N = 3$. By considering the term z^{k-1} that arises from

$$\left[T_{e^{i\theta}r^3}, T_{e^{-2i\theta}f_{-2}}\right]\left(z^k\right) = \left[T_{e^{3i\theta}f_3}, T_{a_4\overline{z}^4}\right]\left(z^k\right) + \left[T_{e^{2i\theta}f_2}, T_{a_3\overline{z}^3}\right]\left(z^k\right) + \left[T_{e^{i\theta}f_1}, T_{a_2\overline{z}^2}\right]\left(z^k\right) + \left[T_{f_0}, T_{a_1\overline{z}}\right]\left(z^k\right), \tag{26}$$

and using Remark 1 (iii) and Lemmas 3, 4, and 5, while assuming that none of the a_4, a_3, a_2, a_1 is zero, we obtain

$$\begin{aligned}\widehat{f_{-2}}(z) = {} & B_{3,-2}\frac{(z+4)}{(z-2)(z)} + a_4C_3\frac{(z+4)}{(z-2)(z)}\sum_{i=0}^{2}\frac{(z+2i-6)(z+2i-4)}{(z+2i+2)(z+2i+4)} \\ & + a_3C_2\frac{(z+4)}{(z-2)(z)}\sum_{i=0}^{1}\frac{(z+2i-4)(z+2i-2)}{(z+2i+2)(z+2i+4)} + \frac{a_2B_{3,1}}{(z+2)} + \frac{a_1a_1C_2}{(z+2)} \\ & + \frac{a_1a_2C_3(z+4)}{(z-2)(z)}\sum_{j=0}^{1}\sum_{i=0}^{1-j}\frac{(z+2i)(z+2i-2+2j)(z+2i-4+2j)}{(z+2i+4)(z+2i+2+2j)(z+2i+2j)} \\ & + \frac{a_1a_2C_3(z+4)}{(z-2)(z)}\sum_{j=0}^{1}\sum_{i=0}^{1-j}\frac{(z+2i)(z+2i-2)(z+2i-4+2j)}{(z+2i+4)(z+2i+2)(z+2i+2j)},\end{aligned} \tag{27}$$

for some constant $B_{3,-2}$. Applying partial fraction decomposition to the term on the right-hand side of equation (27) and using Remark 1 (i) and (ii), we observe that the radial function f_{-2} contains the term $(3B_{3,-2} + a_4C_3)r^{-2}$. This term belongs to $L^1([0,1), rdr)$ only if $3B_{3,-2} = -a_4C_3$. Now, by taking $k = 1$ in equation (26), we obtain

$$\widehat{f_{-2}}(4) = a_4C_3\frac{1}{15} + a_3C_2\frac{1}{10} + a_2B_{3,1}\frac{1}{6} + \frac{1}{6}a_1a_1C_2 + \left(\frac{1}{5} + \frac{1}{18} + \frac{1}{12}\right)a_1a_2C_3. \tag{28}$$

On the other hand, equation (27) implies that

$$\widehat{f_{-2}}(4) = B_{3,-2} + a_4C_3\frac{1}{15} + a_3C_2\frac{1}{10} + a_2B_{3,1}\frac{1}{6} + \frac{1}{6}a_1a_1C_2 + \left(\frac{1}{5} + \frac{1}{18} + \frac{1}{12}\right)a_1a_2C_3. \tag{29}$$

Thus, we must have $B_{3,-2} = 0$. Since $3B_{3,-2} = -a_4C_3$, and we are assuming that $a_4 \neq 0$, it follows that $C_3 = 0$. Hence, $N \leq 2$. □

Proposition 4. *Assume the hypotheses of Theorem 2. If equation (8) holds, then $N \leq 1$. Moreover, under the assumption that $N \geq 1$, we conclude that $N = 1$.*

Proof 6. Suppose $N = 2$. In this case, Part "(i)" of Lemma 5 implies that

$$\begin{aligned}\widehat{f_{N-3}}(z+N-1) &= \widehat{f_{-1}}(z+1) \\ &= B_{2,-1}\frac{z+4}{z(z+2)} + \frac{z-2}{(z+2)(z+2)} + \frac{a_2C_2}{z+6} + \frac{a_1C_1}{z+2}.\end{aligned} \tag{30}$$

Now, by taking $k = 0$ in equation (17), we obtain

$$\widehat{f_{-1}}(3) = a_2 C_2 \frac{1}{8} + a_1 C_1 \frac{1}{4}. \tag{31}$$

On the other hand, equation (30) yields

$$\widehat{f_{-1}}(3) = B_{2,-1} \frac{3}{4} + a_2 C_2 \frac{1}{8} + a_1 C_1 \frac{1}{4}. \tag{32}$$

Thus, we must have $B_{2,-1} = 0$. Next, we analyze the term z^{k-1}, which arises from

$$\left[T_{e^{i\theta} r^3}, T_{e^{-2i\theta} f_{-2}}\right]\left(z^k\right) = \left[T_{e^{2i\theta} f_2}, T_{a_3 \bar{z}^3}\right]\left(z^k\right) + \left[T_{e^{i\theta} f_1}, T_{a_2 \bar{z}^2}\right]\left(z^k\right) + \left[T_{f_0}, T_{a_1 \bar{z}}\right]\left(z^k\right). \tag{33}$$

Assuming that none of the a_3, a_2, a_1 is zero, Remark 1 (iii) and Lemmas 3 and 4 imply that

$$\widehat{f_{-2}}(z) = B_{2,-2} \frac{(z+4)}{(z-2)(z)} + \frac{a_3 C_2 (z+4)}{(z-2)(z)} \sum_{i=0}^{1} \frac{(z+2i-2)(z+2i-4)}{(z+2i+2)(z+2i+4)} + \frac{a_2 C_1}{(z+2)} + \frac{a_1 a_1 C_2}{(z+2)}, \tag{34}$$

for some constant $B_{2,-2}$. By applying partial fraction decomposition to the term on the right-hand side of the equation above and using Remark 1 (i) and (ii), we see that the radial function f_{-2} contains the term $3B_{2,-2} r^{-2}$. Therefore, for f_{-2} to be in $L^1([0,1), rdr)$, we must have $B_{2,-2} = 0$. Finally, we consider the term z^{k-3}, which comes from

$$\left[T_{e^{i\theta} r^3}, T_{e^{-4i\theta} f_{-4}}\right]\left(z^k\right) = \left[T_{e^{2i\theta} f_2}, T_{a_5 \bar{z}^5}\right]\left(z^k\right) + \left[T_{e^{i\theta} f_1}, T_{a_4 \bar{z}^4}\right]\left(z^k\right) + \left[T_{f_0}, T_{a_3 \bar{z}^3}\right]\left(z^k\right) + \left[T_{e^{-i\theta} f_{-1}}, T_{a_2 \bar{z}^2}\right]\left(z^k\right) + \left[T_{e^{-2i\theta} f_{-2}}, T_{a_1 \bar{z}}\right]\left(z^k\right). \tag{35}$$

Assuming none of the a_5, a_4, a_3, a_2, a_1 is zero, Remark 1 (iii) and Lemmas 3, 4, and 5 imply that

$$\widehat{f_{-4}}(z-2) = B_{2,-4} \frac{(z+4)}{(z-6)(z-4)} + a_5 C_2 \frac{(z+4)}{(z-6)(z-4)} \sum_{i=0}^{1} \frac{(z+2i-8)(z+2i-6)}{(z+2i+2)(z+2i+4)} + \frac{a_4 C_1}{(z+2)} + 2\frac{a_1 a_3 C_2}{(z+2)} + \frac{a_2 a_2 C_2}{(z+2)}, \tag{36}$$

for some constant $B_{2,-4}$. By applying partial fraction decomposition to the term on the right-hand side of the equation above and using Remark 1 (i) and (ii), we clearly see that the radial function f_{-4} contains the terms $5B_{2,-4} r^{-4}$ and $(-4B_{2,-4} - (2/3) a_5 C_2) r^{-2}$. Therefore, for f_{-4} to be in $L^1([0,1), rdr)$, we must have $B_{2,-4} = 0$ and $-4B_{2,-4} - (2/3) a_5 C_2 = 0$. Consequently, $a_5 C_2 = 0$. Since we are assuming $a_5 \neq 0$, it follows that $C_2 = 0$. Hence, $N \leq 1$. □

The following lemma will be used to explicitly determine the radial function f_n in the symbol f, for $n \leq -1$.

Lemma 6. *Let $l \in \mathbb{N}$. Assume*

$$\left[T_{e^{i\theta} r^3}, T_{e^{-il\theta} f_{-l}}\right]\left(z^k\right) = \left[c T_{e^{i\theta} r^3}, T_{a_l \bar{z}^l}\right]\left(z^k\right), \tag{37}$$

for some constant c. Then, $T_{e^{-il\theta} f_{-l}} = c T_{a_l \bar{z}^l}$.

Proof 7. If equation (37) holds, then for all $k \geq l$, we have

$$\begin{aligned}(2k-2l+2)\widehat{f_{-l}}(2k-l+2)\frac{(2k-2l+4)}{(2k-2l+6)} - \frac{(2k+4)}{(2k+6)}(2k-2l+4)\widehat{f_{-l}}(2k-l+4) \\ = ca_l\frac{(2k-2l+2)}{(2k+2)}\frac{(2k-2l+4)}{(2k-2l+6)} - ca_l\frac{(2k+4)}{(2k+6)}\frac{(2k-2l+4)}{(2k+4)}.\end{aligned} \tag{38}$$

We complexify the equation above by taking $z = 2k$. Using Remark 1 (ii), we multiply both sides by $((z-2l+6)/(z+4))$ to obtain

$$\begin{aligned}(z-2+2)\widehat{f_{-l}}(z-l+2)\frac{(z-2l+4)}{(z+4)} - \frac{(z-2l+6)}{(z+6)}(z-2l+4)\widehat{f_{-l}}(z-l+4) \\ = ca_l\frac{(z-2l+2)(z-2l+4)}{(z+2)(z+4)} - ca_l\frac{(z-2l+4)(z-2l+6)}{(z+4)(z+6)}.\end{aligned} \tag{39}$$

Using Remark 1 (iii), we find that there exists a constant A_l such that

$$\widehat{f_{-l}}(z-l+2) = A_l\frac{(z+4)}{(z-2l+4)(z-2l+2)} + ca_l\frac{1}{(z+2)}. \tag{40}$$

We now consider the following cases.

Case $l \geq 2$: By applying partial fraction decomposition to the term on the right-hand side of equation (11) and using Remark 1 (i), we see that the radial function f_{-l} contains the term $A_l(l+1)r^{-l}$. Since $-l \leq -2$, the function f_{-l} belongs to $L^1([0,1), rdr)$ only if $A_l = 0$. Hence, $f_{-l}(r) = ca_l r^l$.

Case $l = 1$: By taking $k = 0$ in equation (37), we obtain

$$\widehat{f_1}(3) = ca_1\frac{1}{4}. \tag{41}$$

On the other hand, equation (11) implies that

$$\widehat{f_1}(3) = A_1\frac{3}{4} + ca_1\frac{1}{4}. \tag{42}$$

Therefore, we conclude that $A_1 = 0$, which implies that $f_{-1}(r) = ca_1 r$. □

Finally, we reach the last step of the proof, which is to reconstruct the symbol f.

Proposition 5. *Under the assumptions of Theorem 2, if equation (8) holds, then $f(re^{i\theta}) = C_1 g(re^{i\theta}) + C_0$, for some constants C_1 and C_0.*

Proof 8. If equation (8) holds, then Proposition 4 implies that

$$f\left(re^{i\theta}\right) = \sum_{n=-\infty}^{1} e^{in\theta} f_n. \tag{43}$$

From Lemma 3, we have that $f_1(r) = C_1 r^3$ and $f_0 = C_0$. Next, for all integers $l \geq 1$, the term z^{k-l} comes only from

$$\left[T_{e^{i\theta}r^3}, T_{e^{-il\theta}f_{-l}}\right]\left(z^k\right) = \left[C_1 T_{e^{i\theta}r^3}, T_{a_l\bar{z}^l}\right]\left(z^k\right). \tag{44}$$

Thus, Lemma 6 implies that $f_{-l} = C_1 r^l$. Finally, by combining those results, we obtain

$$\begin{aligned} f\left(re^{i\theta}\right) &= e^{i\theta} f_1(r) + f_0(r) + \sum_{n=-\infty}^{-1} e^{in\theta} f_n(r) \\ &= C_1 e^{i\theta} r^3 + C_0 + \sum_{l=1}^{\infty} C_1 a_l e^{-il\theta} r^l \\ &= C_1 g\left(re^{i\theta}\right) + C_0. \end{aligned} \tag{45}$$

Using the fact that Toeplitz operators are linear with respect to their symbols, we conclude that

$$T_f = C_1 T_g + C_0 I. \tag{46}$$

□

Remark 3. As a direct consequence of Theorem 2, if f and $\tilde{f}$ are two symbols of the form given in the theorem such that both T_f and $T_{\tilde{f}}$ commute with T_g, then T_f and $T_{\tilde{f}}$ must commute with each other. This result aligns with the conjecture stated in ([6], Concluding remarks, p.263), which asserts that if two Toeplitz operators commute with a third one, provided none of them is the identity or the zero operator, then they must commute with each other.

6 | Special Case

In this section, we consider the case where the infinite anti-analytic power series in the symbol g from Theorem 2 is replaced by a polynomial in $\bar{z}$.

Lemma 7. *Let* $g(re^{i\theta}) = e^{i\theta}r^3 + \sum_{l=1}^{m} a_l e^{-il\theta} r^l$. *Then, the product* T_g^2 *is a Toeplitz operator if and only if* $m \leq 4$.

Proof 9. We have

$$T_g^2 = T_{e^{i\theta}r^3} T_{e^{i\theta}r^3 + \sum_{l=1}^{m} a_l e^{-il\theta} r^l} T_{e^{i\theta}r^3 + \sum_{l=1}^{m} a_l e^{-il\theta} r^l}$$

$$= T^2_{e^{i\theta}r^3} + \sum_{l=1}^{m} a_l T_{e^{i\theta}r^3} T_{\bar{z}^l} + \sum_{l=1}^{m} a_l T_{\bar{z}^l} T_{e^{i\theta}r^3} + \sum_{l=1}^{m} a_l T_{\bar{z}^l} T_{\bar{z}^l}$$

$$= T^2_{e^{i\theta}r^3} + \sum_{l=1}^{m} a_l T_{e^{i\theta}r^3} T_{\bar{z}^l} + \sum_{l=1}^{m} a_l T_{\bar{z}^l} T_{e^{i\theta}r^3} + \sum_{l=1}^{m} a_l T_{\bar{z}^{2l}}. \tag{47}$$

It is well-known (see [[22], Example 3.8, p.1057]) that $T^n_{e^{i\theta}r^3}$ is always a Toeplitz operator for all $n \geq 1$. Because $\bar{z}^l$ is an anti-analytic symbol, $T_{\bar{z}^l} T_{e^{i\theta}r^3} = T_{e^{i(\theta - l)}r^{3+l}}$ for all $l \geq 1$. Thus, we focus on the product $T_{e^{i\theta}r^3} T_{\bar{z}^l}$. For all $k \geq l$, we have

$$\begin{aligned} T_{e^{i\theta}r^3} T_{\bar{z}^l}\left(z^k\right) &= \frac{(2k-2l+2)}{(2k+2l+2)} \frac{(2k-2l+4)}{(2k-2l+6)} z^{k+1-l} \\ &= (2k-2l+4)\widehat{h}(2k-l+3) z^{k+1-l}, \end{aligned} \tag{48}$$

where the radial function h is defined as follows:

- $h(r) = (l/(l-1))r^{3l-1} + (1/(1-l))r^{3-l}$ if $l \geq 2$. In this case, h belongs to $L^1([0,1), \mathrm{rdr})$ if and only if $3 - l + 1 \geq 0$, which simplifies to $l \leq 4$.
- $h(r) = r^2 + 4r^2 \log r$ if $l = 1$. Here, h is in $L^1([0,1), \mathrm{rdr})$.

Therefore, for $m \leq 4$, T_g^2 remains a bounded Toeplitz operator, thus completing the proof. □

The above lemma allows us to replace the antianalytic part of the symbol g in Theorem 2 by a polynomial in $\bar{z}$ of degree at most 4, yielding the following result. We provide a brief proof, as the calculations are nearly identical to those in the proof of our main theorem in the previous sections.

Theorem 3. *Let g be a symbol of the form* $g(re^{i\theta}) = e^{i\theta}r^3 + \sum_{l=1}^{m} a_l \bar{z}^l$, *where* $z = re^{i\theta}$, $a_l \in \mathbb{C}$ *and* $m \leq 4$. *If there exists a nonzero function f of the form* $f(re^{i\theta}) = \sum_{n=-\infty}^{N} e^{in\theta} f_n(r)$, *with* $N \geq 1$, *such that the commutator* $[T_f, T_g] = 0$, *then* $T_f = P(T_g)$, *where P is a polynomial of degree at most two.*

Proof 10. Following the same argument as in the previous sections, we deduce that if $[T_f, T_g] = 0$, then $N \leq 2$. Finally, using the same techniques as in the proof of ([7], Theorem 2, p. 886), we conclude that T_f is a polynomial of degree at most two in T_g. □

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